Section 7.5 Inner Product Spaces

With the "dot product" defined in Chapter 6, we were able to study the following properties of vectors in $\mathcal{R}^n$.

1) Length or norm of a vector **u**. ($\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$)
2) Distance of two given vectors **u** and **v**. ($\|\mathbf{u} - \mathbf{v}\|$)
3) Orthogonality of two vectors **u** and **v**. (whether $\mathbf{u} \cdot \mathbf{v} = 0$)

Now, how do we describe the same properties of vectors in other types of vector spaces?

For example,

1) How do we define the norm of the function $f(x)$ in $C([a, b])$?
2) How do we determine whether the polynomials $f(x)$ and $g(x)$ in $\mathcal{P}_2$ are orthogonal?
3) How do we calculate the distance between the two matrices A and B in $\mathcal{M}_{4x3}$?

Section 7.5 Inner Product Spaces

Definition.

Let V be a vector space over a field $\mathcal{F}$ (which is either $\mathcal{R}$ or $\mathcal{C}$). An **inner product** on V is a function that assigns a scalar in $\mathcal{F}$ to any pair of vectors $\mathbf{u}$ and $\mathbf{v}$, denoted $\langle \mathbf{u}, \mathbf{v} \rangle$, such that, for any vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and any scalar c, the following axioms hold.

Axioms of an Inner Product

1. $\langle \mathbf{u}, \mathbf{u} \rangle \in \mathcal{R}$ and $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ if $\mathbf{u} \neq \mathbf{0}$.
2. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$.
3. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.
4. $\langle c\mathbf{u}, \mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$.

Property: Many inner products may be defined on a vector space.

Proof If $\langle \cdot, \cdot \rangle$ is an inner product, then $\langle \cdot, \cdot \rangle_r$ defined by $\langle \mathbf{u}, \mathbf{v} \rangle_r = r\langle \mathbf{u}, \mathbf{v} \rangle$ is also an inner product for any $r > 0$.

Definition.

A vector space endowed with a particular **inner product** is called an **inner product space**.

Example: The dot product is an inner product on $\mathcal{R}^n$.

Definition.

Let V be a vector space over a field $\mathcal{F}$ (which is either $\mathcal{R}$ or $\mathcal{C}$). An **inner product** on V is a function that assigns a scalar in $\mathcal{F}$ to any pair of vectors $\mathbf{u}$ and $\mathbf{v}$, denoted $\langle \mathbf{u}, \mathbf{v} \rangle$, such that, for any vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and any scalar c, the following axioms hold.

Axioms of an Inner Product

1. $\langle \mathbf{u}, \mathbf{u} \rangle \in \mathcal{R}$ and $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ if $\mathbf{u} \neq \mathbf{0}$.
2. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$.
3. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.
4. $\langle c\mathbf{u}, \mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$.

Example: $V = \mathbf{C}([a, b]) = \{f \mid f : [a, b] \to \mathcal{R},\ f \text{ is continuous}\}$ is a vector space, and the function $\langle \cdot, \cdot \rangle : V \times V \to \mathcal{R}$ defined by

$$\langle f, g \rangle = \int_a^b f(t)g(t)dt$$

$\forall f, g \in V$ is an inner product on V.

Axiom 1: f^2 is continuous and non-negative.

$$f \neq \mathbf{0} \Rightarrow f^2(t_0) > 0 \text{ for some } t_0 \in [a, b].$$

$$\Rightarrow f^2(t) > p > 0\ \forall\ [t_0 - r/2,\ t_0 + r/2] \subseteq [a, b].$$

$$\Rightarrow \langle f, f \rangle = \int_a^b f^2(t)dt \geq r \cdot p > 0.$$

Axioms 2 - 4: You examine them.

Example: $\langle A, B\rangle = \text{trace}(AB^T)$ is the Frobenius inner product on $\mathcal{R}^{n\times n}$.

Axiom 1: $\langle A, A\rangle = \text{trace}(AA^T) = \Sigma_{1\le i,j\le n}(a_{ij})^2 > 0\ \forall\ A \neq O$.

Axiom 2: $\text{trace}(AB^T) = \text{trace}(AB^T)^T = \text{trace}(BA^T)$.

Axioms 3 - 4: You examine them.

Definition.

For any vector $\mathbf{v}$ in an **inner product space** V, the **norm** or **length** of $\mathbf{v}$ is denoted and defined as $||\mathbf{v}|| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$. The **distance** between $\mathbf{u}, \mathbf{v} \in V$ is defined as $||\mathbf{u} - \mathbf{v}||$.

Example: The Frobenius norm in $\mathcal{R}^{n\times n}$ is $||A|| = [\Sigma_{1\le i,j\le n}(a_{ij})^2]^{1/2}$, since $\langle A, B \rangle = \text{trace}(AB^T) = \Sigma_{1\le i,j\le n} a_{ij} b_{ij}$.

Property: Inner products and norms satisfy the elementary properties stated in Theorem 6.1, the Cauchy-Schwarz inequality, and the triangle inequality in Section 6.1.

Proof You show it.

Theorem 6.1

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{R}^n$ and c be a scalar in $\mathcal{R}$.

(a) $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$.

(b) $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

(c) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

(d) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

(e) $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$.

(f) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{u}) = \mathbf{u} \cdot (c\mathbf{u})$.

(g) $||c\mathbf{u}|| = |c|||\mathbf{u}||$.

Let V be an inner product space and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V and c be a scalar in $\mathcal{R}$. It can be shown that

(a) $\langle \mathbf{u}, \mathbf{u} \rangle = ||\mathbf{u}||^2$.

(b) $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

(c) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

(d) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.

(e) $\langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle$.

(f) $\langle c\mathbf{u}, \mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{u}, c\mathbf{u} \rangle$.

(g) $||c\mathbf{u}|| = |c|||\mathbf{u}||$.

Theorem 6.2 (Pythagorean theorem in $\mathcal{R}^n$)

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{R}^n$. Then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.$$

Proof

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + \underbrace{2\mathbf{u}\cdot\mathbf{v}}_{\substack{= 0 \text{ if and only if} \\ \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal}}} + ||\mathbf{v}||^2$$

Pythagorean theorem in any inner product space *V*

Let V be an **inner product space** and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V. Then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.$$

Proof

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + 2\underbrace{\langle \mathbf{u},\mathbf{v}\rangle}_{\substack{= 0 \text{ if and only if} \\ \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal}}} + ||\mathbf{v}||^2$$

Theorem 6.3 (Cauchy-Schwarz inequality)

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$, we have

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||.$$

Proof Using Theorem 6.2.

Cauchy-Schwarz inequality in any inner product space *V*

Let V be an **inner product space**. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V, we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||.$$

Proof Using Pythagorean Theorem.

Example: A Cauchy-Schwarz inequality in C($[a, b]$)

$$\left(\int_a^b f(t)g(t)dt \right)^2 \leq \left(\int_a^b f^2(t)dt \right) \left(\int_a^b g^2(t)dt \right)$$

Theorem 6.4 (Triangle inequality)

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$, we have

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||.$$

Proof Using Theorem 6.3.

Triangle inequality in any inner product space *V*

Let V be an **inner product space**. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V, we have

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||.$$

Proof Using Cauchy-Schwartz inequality.

Definition.

In an inner product space V, the vectors $\mathbf{u}$, $\mathbf{v}$ are called **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, a vector $\mathbf{u}$ is called a **unit vector** if $||\mathbf{u}|| = 1$, a subset S is called **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all distinct $\mathbf{u}, \mathbf{v} \in S$, and S is called **orthonormal** if S is **orthogonal** and $||\mathbf{u}|| = 1$ for all $\mathbf{u} \in S$.

Properties:

1. Every nonzero vector $\mathbf{v}$ in an inner product space may be changed into a unit normalized vector $(1/||\mathbf{v}||)\mathbf{v}$, and every orthogonal subset with only nonzero vectors may be changed into an orthonormal subset without affecting the subspace spanned.
2. An orthogonal set of nonzero vectors is L.I., no matter the set is finite or infinite.

Example: In the inner product space C([0, 2π]), the vectors $f(t) = \sin 3t$ and $g(t) = \cos 2t$ are orthogonal, since

$$\langle f, g \rangle = \int_0^{2\pi} \sin 3t \cos 2t dt = \frac{1}{2} \int_0^{2\pi} [\sin 5t + \sin t] \, dt = 0$$

Example: In the vector space of trigonometric polynomials

$$\mathcal{T}[0, 2\pi] = \text{Span } \{1, \cos t, \sin t, \cos 2t, \sin 2t, \cdots, \cos nt, \sin nt, \cdots\}$$

$= \text{Span } S \Rightarrow S$ is orthogonal, since

$$\langle \cos nt, \sin mt \rangle = \int_0^{2\pi} \cos nt \sin mt dt = 0, \forall n, m \geq 0$$

$$\langle \cos nt, \cos mt \rangle = \int_0^{2\pi} \cos nt \cos mt dt = 0, \forall n \neq m$$

$$\langle \sin nt, \sin mt \rangle = \int_0^{2\pi} \sin nt \sin mt dt = 0, \forall n \neq m$$

$\Rightarrow S$ is a basis of $\mathcal{T}$ $[0, 2\pi]$.

Theorem 6.5 (Gram-Schmidt Process)

Let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ be a basis for a subspace W of $\mathcal{R}^n$. Define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1, \\
&\vdots \\
\mathbf{v}_k &= \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_k \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 - \cdots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{||\mathbf{v}_{k-1}||^2}\mathbf{v}_{k-1}.
\end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_i\}$ is an orthogonal set of nonzero vectors such that

$$\text{Span }\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_i\} = \text{Span }\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_i\}$$

for each i. So $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is an **orthogonal basis** for W.

Gram-Schmidt Process for any inner product space

Let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ be a basis for an inner product space V. Define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1, \\
&\vdots \\
\mathbf{v}_k &= \mathbf{u}_k - \frac{\langle \mathbf{u}_k, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_k, \mathbf{v}_2 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_2 - \cdots - \frac{\langle \mathbf{u}_k, \mathbf{v}_{k-1} \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_{k-1}.
\end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_i\}$ is an orthogonal set of nonzero vectors such that

$$\text{Span } \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_i\} = \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_i\}$$

for each i. So $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is an **orthogonal basis** for W.

Proposition: The Gram-Schmidt process is valid for any inner product space.

Proof You show it.

Corollary: Every finite-dimensional inner product space has an orthonormal basis.

Example: $\mathscr{P}_2$ is an inner product space with the following inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$

$\forall f, g \in \mathscr{P}_2$. From a basis $\mathscr{B} = \{1, x, x^2\} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathscr{P}_2$, an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ may be obtained by applying the Gram-Schmidt process.

Example: $\mathscr{P}_2$ is an inner product space with the following inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$

$\forall f, g \in \mathscr{P}_2$. From a basis $\mathscr{B} = \{1, x, x^2\} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathscr{P}_2$, an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ may be obtained by applying the Gram-Schmidt process.

$$\mathbf{v}_1 = \mathbf{u}_1 = 1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 = x - \frac{\int_{-1}^{1} t \cdot 1dt}{\int_{-1}^{1} 1^2 dt}(1) = x - 0 \cdot 1 = x$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 = x^2 - \frac{\int_{-1}^{1} t^2 \cdot 1dt}{\int_{-1}^{1} 1^2 dt}(1) - \frac{\int_{-1}^{1} t^2 \cdot tdt}{\int_{-1}^{1} t^2 dt}(x)$$

$$= x^2 - \frac{\frac{2}{3}}{2} \cdot 1 - 0 \cdot x = x^2 - \frac{1}{3}.$$

To get an orthonormal basis, compute

$$||\mathbf{v}_1|| = \sqrt{\int_{-1}^{1} 1^2 dx} = \sqrt{2} \qquad ||\mathbf{v}_2|| = \sqrt{\int_{-1}^{1} x^2 dx} = \sqrt{\frac{2}{3}}$$

$$||\mathbf{v}_3|| = \sqrt{\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx} = \sqrt{\frac{8}{45}}$$

and get

$$\left\{ \frac{1}{||\mathbf{v}_1||}\mathbf{v}_1, \frac{1}{||\mathbf{v}_2||}\mathbf{v}_2, \frac{1}{||\mathbf{v}_3||}\mathbf{v}_3, \right\} = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\}$$

For $\mathscr{P}$ with the same inner product and the basis $\mathscr{B} = \{1, x, x^2, \cdots\}$, the same procedure may be applied to obtain an orthonormal basis $\{p_0(x), p_1(x), p_2(x), \cdots\}$, called the normalized Legendre polynomials. Note that $p_0(x)$, $p_1(x)$, and $p_2(x)$ are the above three orthonormal ones.

Proposition

Suppose that V is an inner product space and at W is a finite-dimensional subspace of V. For every $\mathbf{v}$ in V, there exist unique $\mathbf{w} \in W$ and $\mathbf{z} \in W^{\perp}$ such that $\mathbf{v} = \mathbf{w} + \mathbf{z}$. The vector $\mathbf{v}$ is called the **orthogonal projection of v onto** W, and we have

$$\mathbf{w} = \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{v}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{v}, \mathbf{v}_n \rangle \mathbf{v}_n$$

if $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis of W.

Proof You show that the proof of Theorem 6.7 in Section 6.3 is also applicable here.

Corollary: Under the notations in the above Proposition, among all vectors in W, the vectors closest to $\mathbf{v}$ is $\mathbf{w}$.

Proof Follow the derivations of the closest vector property in Section 6.3.

Since the closeness is measured by the distance, which involves the sum (integral) of a square of the difference vector (function), the closest vector is called the **least-square approximation**.

Example: $\mathscr{P}_2$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$ for all $f, g \in \mathscr{P}_2$ is a finite-dimensional subspace of C([-1, 1]).

To $\mathbf{v} = f(x) = \sqrt[3]{x} \in$ C([-1, 1]), the least-squares approximation by a polynomial with degree ≤ 2 is the orthogonal projection of f onto $\mathscr{P}_2$.

odd function

Thus take the orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$,

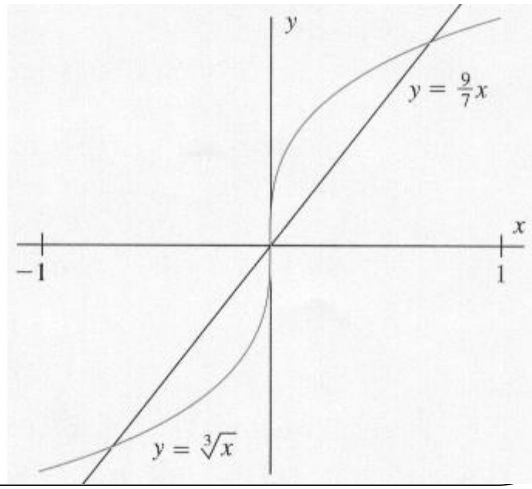
where $\mathbf{v}_1 = \frac{1}{\sqrt{2}}$ $\quad \mathbf{v}_2 = \sqrt{\frac{3}{2}}x$ $\quad \mathbf{v}_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$

even function

and get

$$\begin{aligned} \mathbf{w} &= \langle \mathbf{v}, \mathbf{v}_1\rangle \mathbf{v}_1 + \langle \mathbf{v}, \mathbf{v}_2\rangle \mathbf{v}_2 + \langle \mathbf{v}, \mathbf{v}_3\rangle \mathbf{v}_3 \\ &= \left(\int_{-1}^{1} \sqrt[3]{x} \cdot \sqrt{\frac{3}{2}}x dx\right) \sqrt{\frac{3}{2}}x \\ &= \frac{9}{7}x \end{aligned}$$

0 0



Definition.
A function $y = f(t)$ is called **periodic of period p** if $f(t) = f(t+p)$ for all t.

The least-squares approximation by the trigonometric polynomials of a continuous periodic function $f(t)$ of period 2π.
Periodicity $\Rightarrow$ consider the approximation over a period $[0, 2\pi]$.

$\Rightarrow f(\cdot) \in C([0, 2\pi])$, and can be orthogonally projected onto a subspace W_n spanned by an orthogonal set

$$S_n = \{1, \cos t, \sin t, \cos 2t, \sin 2t, , \ldots, \cos nt, \sin nt\}$$

To have an orthonormal basis for W_n, compute

$$||1|| = \sqrt{\int_0^{2\pi} 1 dt} = \sqrt{2\pi}$$

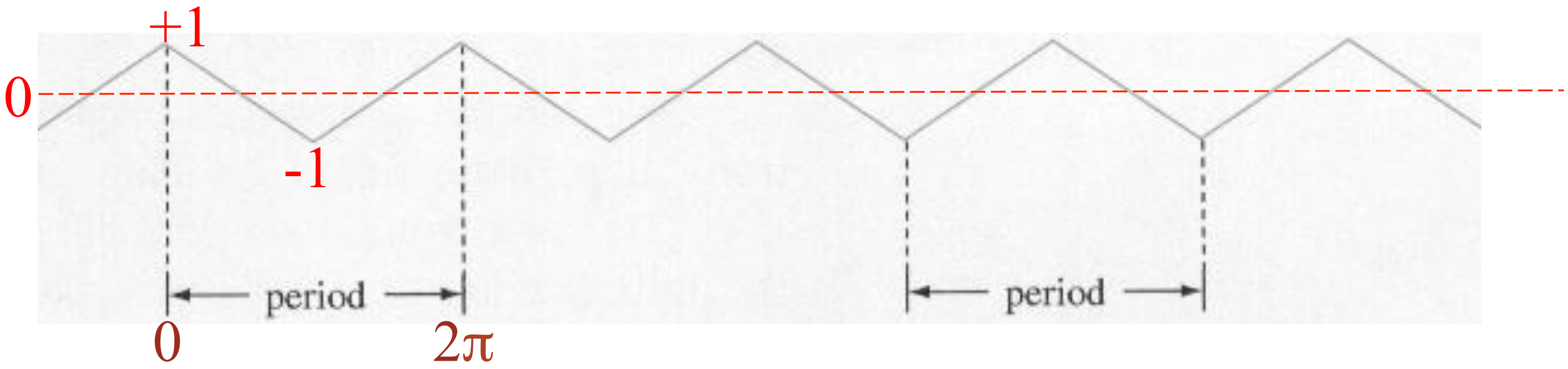
$$||\cos kt|| = \sqrt{\int_0^{2\pi} \cos^2 kt dt} = \sqrt{\frac{1}{2}\int_0^{2\pi} (1 + \cos 2kt) dt} = \sqrt{\pi}$$

$$||\sin kt|| = \sqrt{\pi}$$

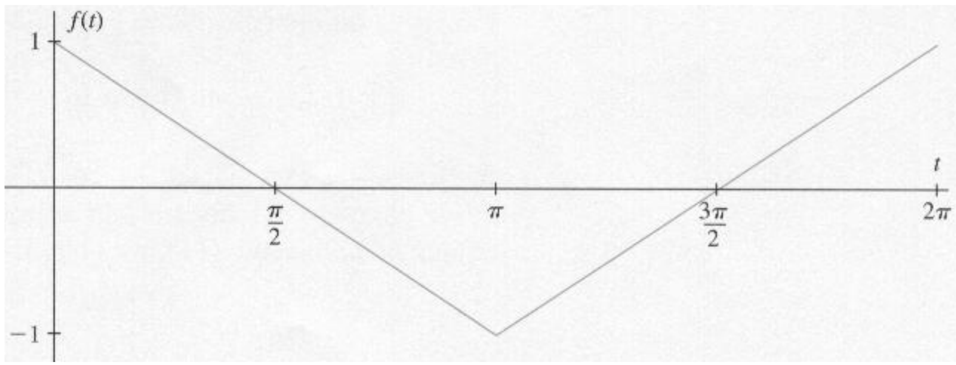
and get the orthonormal basis

$$\mathcal{B}_n = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, , \frac{1}{\sqrt{\pi}} \cos 2t, \frac{1}{\sqrt{\pi}} \sin 2t, \cdots , \frac{1}{\sqrt{\pi}} \cos nt, \frac{1}{\sqrt{\pi}} \sin nt \right\}$$

Suppose that $f(\cdot)$ is the sawtooth function



$$f(t) = \begin{cases} 1 - \frac{2}{\pi} t & \text{if } 0 \leq t \leq \pi \\ \frac{2}{\pi} t - 3 & \text{if } \pi \leq t \leq 2\pi \end{cases}$$



Let f_n be the least-squares approximation of f by W_n (the orthogonal projection of f onto W_n). Then

$$f_n = \left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle \frac{1}{\sqrt{2\pi}} + \left\langle f, \frac{1}{\sqrt{\pi}} \cos t \right\rangle \frac{1}{\sqrt{\pi}} \cos t + \left\langle f, \frac{1}{\sqrt{\pi}} \sin t \right\rangle \frac{1}{\sqrt{\pi}} \sin t + \cdots$$

$$+ \left\langle f, \frac{1}{\sqrt{\pi}} \cos nt \right\rangle \frac{1}{\sqrt{\pi}} \cos nt + \left\langle f, \frac{1}{\sqrt{\pi}} \sin nt \right\rangle \frac{1}{\sqrt{\pi}} \sin nt$$

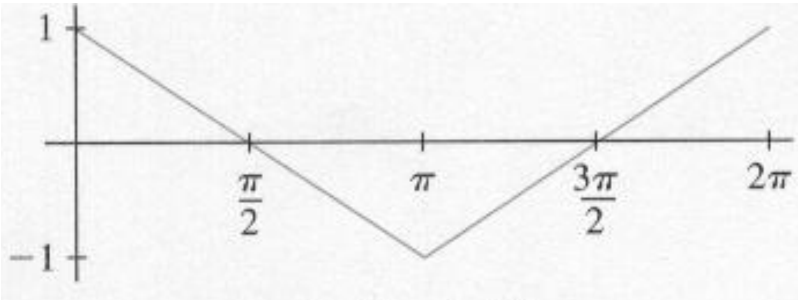
Now,

$$\left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} \left(1 - \frac{2}{\pi} t\right) dt + \frac{1}{\sqrt{2\pi}} \int_{\pi}^{2\pi} \left(\frac{2}{\pi} t - 3\right) dt = 0 + 0$$

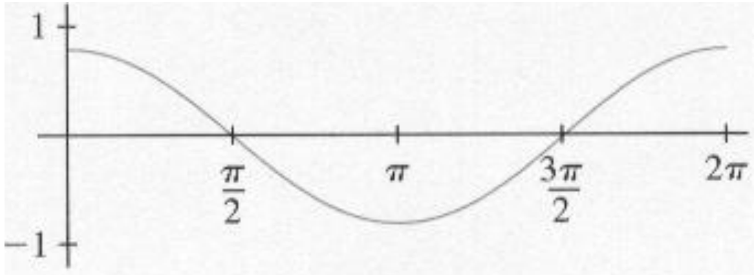
$$\begin{aligned} \left\langle f, \frac{1}{\sqrt{\pi}} \cos kt \right\rangle &= \frac{1}{\sqrt{\pi}} \int_0^{\pi} \left(1 - \frac{2}{\pi} t\right) \cos kt \ dt + \frac{1}{\sqrt{\pi}} \int_{\pi}^{2\pi} \left(\frac{2}{\pi} t - 3\right) \cos kt \ dt \\ &= \frac{4}{\pi\sqrt{\pi}k^2}(1 - (-1)^k) \end{aligned}$$

and $\left\langle f, \frac{1}{\sqrt{\pi}} \sin kt \right\rangle = 0$

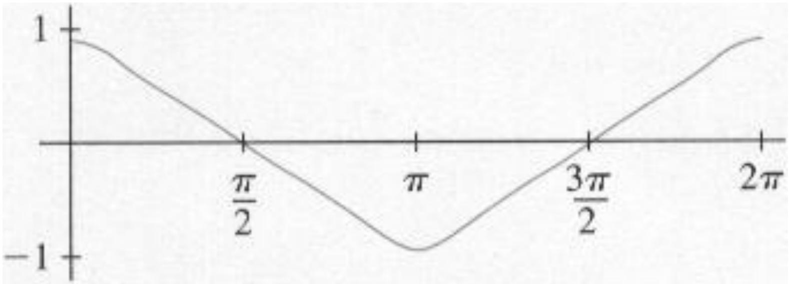
$$\Rightarrow \quad f_n(t) = \frac{8}{\pi^2}\left[\frac{\cos t}{1^2} + \frac{\cos 3t}{3^2} + \cdots + \frac{\cos nt}{n^2}\right]$$



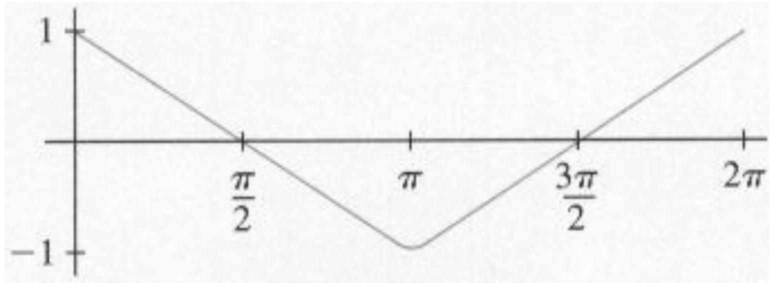
graph of sawtooth tone



graph of f_1



graph of f_5



graph of f_{15}

Homework Set for Section 7.5

- Section 7.5: Problems 1, 4, 9, 13, 17, 45, 46, 51, 53, 60, 62, 63, 64, 71, 75