Section 7.4 Matrix Representations of Linear Operators

Definition.

Let V be a **finite-dimensional** vector space and $\mathcal{B}$ be a basis for V. For any vector $\mathbf{v}$ in V, the vector $\Phi_{\mathcal{B}}(\mathbf{v})$ is called the **coordinate vector** of $\mathbf{v}$ relative to $\mathcal{B}$ and is denoted as $[\mathbf{v}]_{\mathcal{B}}$.

$\Phi_{\mathcal{B}} : V \rightarrow \mathcal{R}^n$ defined as

$$\Phi_{\mathcal{B}}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n) = [c_1\ c_2\ \ldots\ c_n]^T.$$

Property:

$[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}$ and $[c\mathbf{u}]_{\mathcal{B}} = c[\mathbf{u}]_{\mathcal{B}}$ for all $\mathbf{u}, \mathbf{v} \in V$ and scalar c.

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Example: Let V = Span $\mathcal{B}$, where $\mathcal{B} = \{e^t \cos t, e^t \sin t\}$ is L.I. and thus a basis of V.

Consider the function $\mathbf{v} = e^t \cos(t - \pi/4)$. (Is it in V?)

Then $\mathbf{v}$ is in V since

$$\mathbf{v} = \frac{1}{\sqrt{2}} e^t (\cos t + \sin t) = \frac{1}{\sqrt{2}} e^t \cos t + \frac{1}{\sqrt{2}} e^t \sin t$$

In addition, $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$

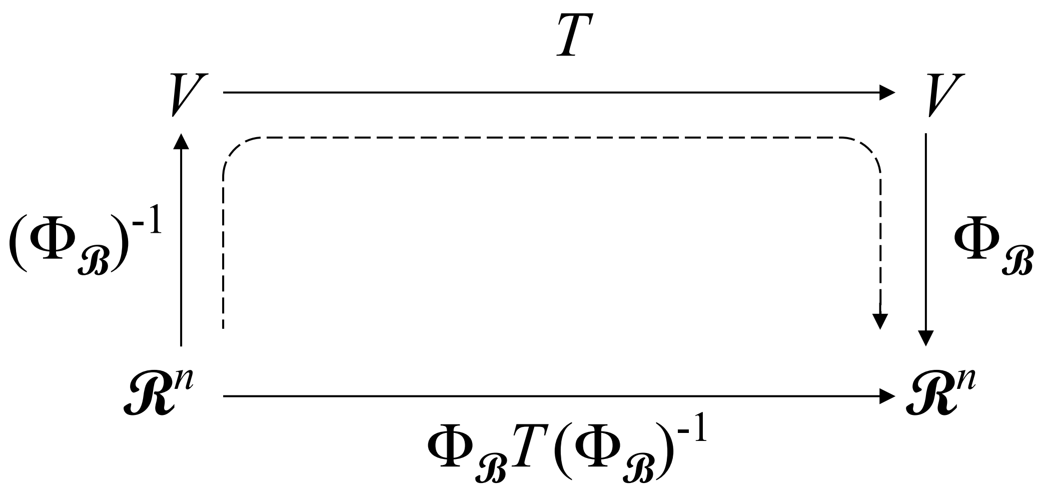
Consider a linear transformation T: $V \to W$

Questions:

1) Can we define a "standard matrix" for T?
2) If not, what kind of matrix representation of T can we formulate?

In this course, we will consider only a simpler case where T is a linear operator (i.e., the domain and the codomain are the same vector space).

Let $T : V \to V$ be a linear operator on an n-dimensional vector space V with a basis $\mathcal{B}$. Define the linear operator $\Phi_{\mathcal{B}} T (\Phi_{\mathcal{B}})^{-1} : \mathcal{R}^n \to \mathcal{R}^n$, and consider its standard matrix A, called the matrix representation of T with respect to $\mathcal{B}$ and denoted as $[T]_{\mathcal{B}}$. With the notations, $[T]_{\mathcal{B}} = A$ and $T_A = \Phi_{\mathcal{B}} T (\Phi_{\mathcal{B}})^{-1}$.



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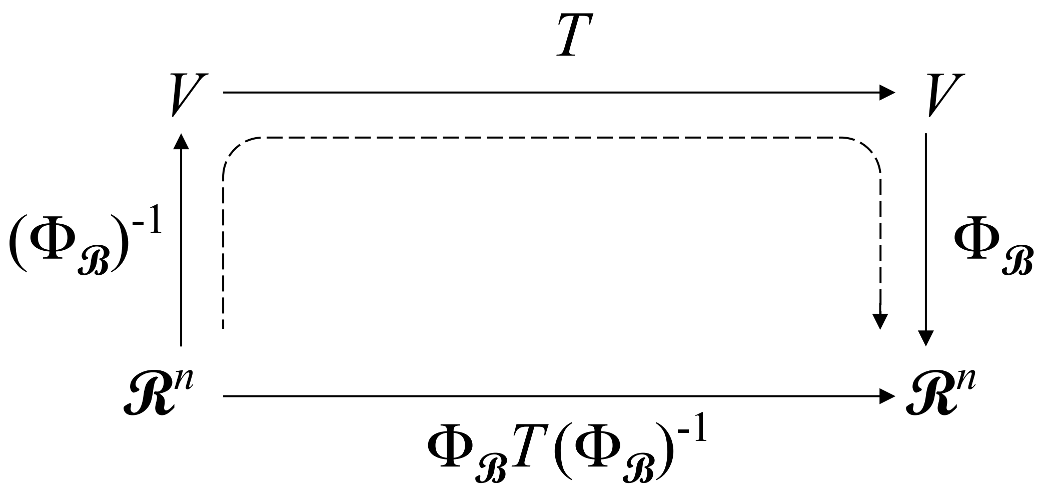

Question: How to express $[T]_{\mathcal{B}}$ in terms of T and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$?

Property:

If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then $[T]_{\mathcal{B}} = [\ [T(\mathbf{v}_1)]_{\mathcal{B}}\ \ [T(\mathbf{v}_2)]_{\mathcal{B}}\ \cdots\ \ [T(\mathbf{v}_n)]_{\mathcal{B}}\ ]$.

Proof $[T]_{\mathcal{B}} = A \Rightarrow A\mathbf{e}_j = T_A(\mathbf{e}_j) = \Phi_{\mathcal{B}}\, T\, (\Phi_{\mathcal{B}})^{-1}(\mathbf{e}_j) = \Phi_{\mathcal{B}}\, T(\mathbf{v}_j) = [T(\mathbf{v}_j)]_{\mathcal{B}}$.

Example: Let $T: \mathcal{P}_2 \rightarrow \mathcal{P}_2$ be defined by $T(p(x)) = p(0) + 3p(1)x + p(2)x^2$ for all $p(x)$ in $\mathcal{P}_2$. Then T is linear. For $\mathcal{B} = \{1, x, x^2\}$, $[T]_{\mathcal{B}} = A = [\ \mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ ]$ and

$$\mathbf{a}_1 = [T(1)]_B = [1 + 3x + x^2]_B = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix},\ \mathbf{a}_2 = [T(x)]_B = [3x + 2x^2]_B = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix},$$

$$\mathbf{a}_3 = [T(x^2)]_B = [3x + 4x^2]_B = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix},\ \text{so } [T]_B = [\mathbf{a}_1\ \ \ \mathbf{a}_2\ \ \ \mathbf{a}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}.$$

Example: Let V = Span $\mathcal{B}$, where $\mathcal{B} = \{e^t \cos t, e^t \sin t\}$ is L.I. and thus a basis of V, and the linear operator $D: V \to V$ be defined by $D(f) = f'$ for all $f \in V$. Then

$$\begin{aligned} D(e^t \cos t) &= (1)e^t \cos t + (-1)e^t \sin t \\ D(e^t \sin t) &= (1)e^t \cos t + (1)e^t \sin t \end{aligned} \Rightarrow [D]_B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Theorem 7.10

Let T be a linear operator on a finite-dimensional vector space V with basis $\mathcal{B}$. Then for any vector $\mathbf{v}$ in V,

$$[T(\mathbf{v})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}.$$

Proof $[T(\mathbf{v})]_{\mathcal{B}} = \Phi_{\mathcal{B}}\, T(\mathbf{v}) = \Phi_{\mathcal{B}}\, T\,(\Phi_{\mathcal{B}})^{-1}\Phi_{\mathcal{B}}(\mathbf{v}) = T_A([\mathbf{v}]_{\mathcal{B}}) = [T]_{\mathcal{B}}\,[\mathbf{v}]_{\mathcal{B}}$,

where $A = [T]_{\mathcal{B}}$.

Example: Relative to the basis $\mathcal{B} = \{1, x, x^2\}$ of $\mathcal{P}_2$, the coordinate vector of $p(x) = 5 - 4x + 3x^2$ is $[p(x)]_{\mathcal{B}} = [\ 5\ \ -4\ \ 3\ ]^T$.

Then $[p'(x)]_{\mathcal{B}} = [D(p(x))]_{\mathcal{B}} = [D]_{\mathcal{B}}\,[p(x)]_{\mathcal{B}}$,where D: $\mathcal{P}_2 \to \mathcal{P}_2$ is defined by $D(p(x)) = p'(x)$, and

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow [p'(x)]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ 0 \end{bmatrix}.$$

The Matrix Representation of the Inverse of an Invertible Linear Operator

Let T be a linear operator on a finite-dimensional vector space V with basis $\mathcal{B}$ and let $A = [T]_{\mathcal{B}}$. Then the following statements are true.
(a) T is invertible if and only if A is invertible.
(b) If T is invertible, then $[T^{-1}]_{\mathcal{B}} = A^{-1}$.

Proof (a) Note that $\Phi_{\mathcal{B}}$ is an isomorphism with an inverse $(\Phi_{\mathcal{B}})^{-1}$, which is also an isomorphism.

If T is invertible, then $T_A = \Phi_{\mathcal{B}} T (\Phi_{\mathcal{B}})^{-1}$ is a composition of isomorphisms. So T_A is invertible and has an invertible standard matrix A.

If A is invertible, then $T_A = \Phi_{\mathcal{B}} T (\Phi_{\mathcal{B}})^{-1}$ is invertible. So $T = (\Phi_{\mathcal{B}})^{-1} T_A \Phi_{\mathcal{B}}$ is invertible.

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(a) T is invertible if and only if A is invertible.
(b) If T is invertible, then $[T^{-1}]_{\mathcal{B}} = A^{-1}$.

Proof (b) By (a) and the invertibility of T, $T_C = \Phi_{\mathcal{B}} T^{-1} (\Phi_{\mathcal{B}})^{-1}$,
where $C = [T^{-1}]_{\mathcal{B}}$.
Also by (a), $T_{A^{-1}} = (T_A)^{-1} = \Phi_{\mathcal{B}} T^{-1} (\Phi_{\mathcal{B}})^{-1}$.
$\Rightarrow T_C = T_{A^{-1}} \Rightarrow C = A^{-1}$.

Example: In the vector space V with a basis $\mathcal{B} = \{e^t \cos t, e^t \sin t\}$ and a linear operator $D: V \to V$ defined by $D(f) = f' \ \ \forall f \in V$,

it is known that $[D]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. So an anti-derivative of

$e^t \sin t$ is $D^{-1}(e^t \sin t)$.
Since $[D^{-1}]_{\mathcal{B}} = ([D]_{\mathcal{B}})^{-1}$ and $[e^t \sin t]_{\mathcal{B}} = [\,0 \ \ 1]^T$,

$$[D^{-1}(e^t \sin t)]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

i.e., $D^{-1}(e^t \sin t) = -(e^t \cos t)/2 + (e^t \sin t)/2$.

Definition.

For a linear operator T on a vector space V (over a field $\mathcal{F}$), a nonzero vector $\mathbf{v}$ in V is said to be an **eigenvector** of T corresponding to the **eigenvalue** λ if there is a scalar $\lambda \in \mathcal{F}$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$. For an eigenvalue λ of T, the set of all vectors $\mathbf{v} \in V$ satisfying $T(\mathbf{v}) = \lambda\mathbf{v}$ is the **eigenspace** of T corresponding to λ.

Example: For linear operator D: $C^\infty \rightarrow C^\infty$ defined by $D(f) = f'$ with $C^\infty = \{f \mid f: \mathcal{R} \rightarrow \mathcal{R},\ f$ has derivatives of all order$\}$, it has an eigenvector $e^{\lambda t} \in C^\infty$ corresponding to the eigenvalue λ, since $D(f)(t) = (e^{\lambda t})' = \lambda e^{\lambda t} = \lambda f(t)$.

$\Rightarrow$ Any scalar λ is an eigenvalue of D.

$\Rightarrow$ D has infinitely many eigenvalues.

Example: f is a solution of $y'' + 4y = 0 \Rightarrow f \in C^\infty$, since f must be twice-differentiable and $f'' = -4f$, which imply that the fourth derivative of f exists ($f'''' = -4f''$), and so on.

$\Rightarrow f \neq 0$ is an eigenvector of $D^2 : C^\infty \to C^\infty$ corresponding to the eigenvalue -4

$\Rightarrow f \in$ eigenspace of D^2 corresponding to the eigenvalue -4

Clearly, every vector in the eigenspace of D^2 corresponding to the eigenvalue -4 is a solution of $y'' + 4y = 0$.

Example: U: $\mathcal{R}^{n\times n} \to \mathcal{R}^{n\times n}$ defined by $U(A) = A^T$ is an isomorphism.

The eigenspace of U corresponding to the eigenvalue 1 is the set of $A \in \mathcal{R}^{n\times n}$ such that $U(A) = A^T = A$, i.e., the set of symmetric matrices.

The eigenspace of U corresponding to the eigenvalue -1 is the set of $A \in \mathcal{R}^{n\times n}$ such that $U(A) = A^T = -A$, i.e., the set of skew-symmetric matrices.

U only has eigenvalues 1 and -1, since $U(A) = A^T = \lambda A$ implies $A = (A^T)^T = (\lambda A)^T = \lambda A^T = \lambda(\lambda A) = \lambda^2 A$.

Eigenvalues and Eigenvectors of a Matrix Representation of a Linear Operator

Let T be a linear operator on a finite-dimensional vector space V with basis $\mathcal{B}$ and let $A = [T]_{\mathcal{B}}$. Then a vector $\mathbf{v}$ in V is an eigenvector of T with corresponding eigenvalue λ if and only if $[\mathbf{v}]_{\mathcal{B}}$ is an eigenvector of A with corresponding eigenvalue λ.

Proof "only if" ($\Rightarrow$) Suppose $\mathbf{v} \neq \mathbf{0}$ satisfies $T(\mathbf{v}) = \lambda\mathbf{v}$.

$$\Rightarrow [\mathbf{v}]_{\mathcal{B}} \neq \mathbf{0} \text{ and } A[\mathbf{v}]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}} = [\lambda\mathbf{v}]_{\mathcal{B}} = \lambda[\mathbf{v}]_{\mathcal{B}}$$

"if" ($\Leftarrow$) Suppose $\mathbf{w} \neq \mathbf{0}$ satisfies $A\mathbf{w} = [T]_{\mathcal{B}}\mathbf{w} = \lambda\mathbf{w}$.

Let $\mathbf{v} = (\Phi_{\mathcal{B}})^{-1}(\mathbf{w}) \neq \mathbf{0}$.

$$\Rightarrow \Phi_{\mathcal{B}}\, T(\mathbf{v}) = [T(\mathbf{v})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = \lambda[\mathbf{v}]_{\mathcal{B}} = \Phi_{\mathcal{B}}(\lambda\mathbf{v})$$

$\Rightarrow T(\mathbf{v}) = \lambda\mathbf{v}$ since $\Phi_{\mathcal{B}}$ is an isomorphism.

Example: Let $T: \mathscr{P}_2 \to \mathscr{P}_2$ be defined as $T(p(x)) = p(0) + 3p(1)x + p(2)x^2$ for all $p(x)$ in $\mathscr{P}_2$. Then T is linear, and

$$[T]_{\mathscr{B}} = A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

where $\mathscr{B} = \{1, x, x^2\}$ is a basis of $\mathscr{P}_2$.

The characteristic polynomial of A is $-(t - 1)^2(t - 6)$.

Span$\{[\ 0\ \ -3\ \ 2\ ]^T\}$ is the eigenspace of A corresponding to the eigenvalue 1 $\Rightarrow$ $ap(x)$ with $a \neq 0$ and $p(x) = -3x + 2x^2$ is an eigenvector of T corresponding to the eigenvalue 1.

Span$\{[\ 0\ \ 1\ \ 1\ ]^T\}$ is the eigenspace of A corresponding to the eigenvalue 6 $\Rightarrow$ $bq(x)$ with $b \neq 0$ and $q(x) = x + x^2$ is an eigenvector of T corresponding to the eigenvalue 6.

Example: U: $\mathcal{R}^{2\times 2} \to \mathcal{R}^{2\times 2}$ defined by $U(A) = A^T$ is an isomorphism and only has eigenvalues 1 and -1.

Let a basis of $\mathcal{R}^{2\times 2}$ be

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\Rightarrow \quad [U]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ The characteristic polynomial of $[U]_{\mathcal{B}}$ is $(t - 1)^3(t + 1)$, for which indeed the only roots are 1 and -1.

Homework Set for Section 7.4

- Section 7.4: Problems 1, 5, 9, 11, 19, 21, 23, 28, 32, 36, 39, 40, 43, 44, 46.