Section 7.3 Basis and Dimension

Definition.

An finite subset $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ of a vector space V is said to be **linearly dependent** if there exist scalars $c_1, c_2, \cdots, c_n$, at least one of which is nonzero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_n\mathbf{v}_n = \mathbf{0}.$$

S is said to be **linearly independent** if S is not **linearly dependent**.

Example: $S = \{x^2 - 3x + 2,\ 3x^2 - 5x,\ 2x - 3\}$, a subset of $\mathscr{P}_2$ is L.D., since

$$3(x^2 - 3x + 2) + (-1)(3x^2 - 5x) + 2(2x - 3) = \mathbf{0}$$

Example: $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$, a subset of $\mathscr{R}^{2\times 2}$ is L.I., since

$$a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

implies that $a = b = c = 0$.

Example: $S = \{e^t, e^{2t}, e^{3t}\}$ is a L.I. subset of $\mathcal{F}(\mathcal{R})$, since if $h(t) = ae^t + be^{2t} + ce^{3t} = 0\ \forall t$, then $h(0) = a + b + c = 0$, $h'(0) = a + 2b + 3c = 0$, and $h''(0) = a + 4b + 9c = 0$ together imply $a = b = c = 0$.

Definition.

An infinite subset S of a vector space V is **linearly dependent** if some finite subset of S is **linearly dependent**. An infinite set S is **linearly independent** if S is not linearly dependent; that is, if every finite subset of S is linearly independent.

Example: The infinite subset $\{1, x, x^2, \cdots, x^n, \cdots\}$ of $\mathscr{P}$ is L.I., since given any nonempty index set I, $\Sigma_{i \in I} c_i x^i = 0$ implies $c_i = 0$ for all $i \in I$. Thus its every nonempty finite subset is L.I..

Example: The infinite subset $\{1+x, 1\text{-}x, 1+x^2, 1\text{-}x^2, \cdots, 1+x^n, 1\text{-}x^n, \cdots\}$ of $\mathscr{P}$ is L.D., since it contains L.D. finite subsets like $\{1+x, 1\text{-}x, 1+x^2, 1\text{-}x^2\}$, in which

$$1(1+x) + 1(1-x) + (-1)(1+x^2) + (-1)(1-x^2) = \mathbf{0}.$$

Theorem 7.8

Let V and W be **vector spaces**, $T : V \to W$ be an **isomorphism**, and $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be a **linear independent** subset of V.
Then the set of images $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \cdots, T(\mathbf{v}_k)\}$ is a **linear independent** subset of W.

Proof $c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots + c_k T(\mathbf{v}_k) = \mathbf{0}$

$\Rightarrow T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = \mathbf{0}$

$\Rightarrow c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ because T is one-to-one

$\Rightarrow c_1 = c_2 = \cdots = c_k = 0.$

Definition.

A subset S of a vector space V is a **basis** of V if S is a **linearly independent** set and a **generating set** of V.

Example: $S = \{1, x, x^2, \cdots, x^n, \cdots\}$ is a basis of $\mathscr{P}$.

It will be shown that if a vector space V has a **finite** basis, then every basis of V is finite and contains the same number of vectors.
It can be shown that every vector space has a basis, provided some axiom of choice from the set theory is accepted.

Theorem 7.9

Let V be a vector space with a finite basis. Then every basis for V is finite and contains the same number of vectors.

Proof Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of V.

Consider a linear transformation $\Phi_{\mathcal{B}} : V \to \mathcal{R}^n$ defined as

$\Phi_{\mathcal{B}}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n) = [c_1\ c_2\ \ldots\ c_n]^T$.

Suppose V has another basis $\mathcal{A}$ (finite of infinite) containing more vectors than $\mathcal{B}$.

$\Rightarrow$ $\mathcal{A}$ has a subset $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n, \mathbf{w}_{n+1}\}$

$\Rightarrow$ S is L.I., since is $\mathcal{A}$ is L.I.

$\Rightarrow$ $\{\Phi_{\mathcal{B}}(\mathbf{w}_1), \Phi_{\mathcal{B}}(\mathbf{w}_2), \ldots, \Phi_{\mathcal{B}}(\mathbf{w}_n), \Phi_{\mathcal{B}}(\mathbf{w}_{n+1})\}$ is L.I. in $\mathcal{R}^n$, since $\Phi_{\mathcal{B}}$ is an isomorphism.

⌧ contradiction, since in $\mathcal{R}^n$ L.I. sets have at most n vectors.

Thus $\mathcal{A}$ can not contain more vectors than $\mathcal{B}$.

Theorem 7.9

Let V be a vector space with a finite basis. Then every basis for V is finite and contains the same number of vectors.

Proof

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Reversing the roles of $\mathcal{A}$ and $\mathcal{B}$, we see that $\mathcal{B}$ can not contain more vectors than $\mathcal{A}$.

$\Rightarrow \mathcal{A}$ and $\mathcal{B}$ contain the same number of vectors.

Definitions.

The zero vector space and vector spaces having a finite basis are called **finite-dimensional**. Others are called **infinite-dimensional**. The dimension of a **finite-dimensional** vector space is equal to the number of vector(s) in any of its bases.

Property: If V is an infinite-dimensional vector space, then V contains an infinite L.I. set.

Proof $V \neq \{\mathbf{0}\} \Rightarrow \exists\, \mathbf{v}_1 \neq \mathbf{0}$ in $V \Rightarrow \exists\, \mathbf{v}_2 \notin \text{Span}\{\mathbf{v}_1\}$ in V (otherwise dim. $V = 1$) $\Rightarrow \exists\, \mathbf{v}_3 \notin \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ in V (otherwise dim. $V = 2$) $\Rightarrow \cdots \Rightarrow \exists\, \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \ldots\}$ in V such that for all n, $\mathbf{v}_{n+1} \notin \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \Rightarrow \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \ldots\}$ is L.I..

Property: Dimension is preserved under isomorphism.

Proof If the vector space V has a finite basis $\mathcal{B}$, and is isomorphic to the other vector space W through an isomorphism T, then you show that the image of $\mathcal{B}$ by T is a basis of W.

$\Rightarrow$ dim. V = dim. W.

You show the infinite-dimensional case.

Properties of finite-dimensional vector spaces:

Suppose V is an n-dimensional vector space.

1. Any L.I. subset of V contains at most n vectors.
2. Any L.I. subset of V that contains exactly n vectors is a basis for V.
3. Any generating set of V contains at least n vectors.
4. Any generating set of V that contains exactly n vectors is a basis for V.

Proof Refer to $\mathcal{R}^n$ via $\Phi_{\mathcal{B}}$.

Example: $\mathcal{B} = S_n = \{1, x, x^2, \cdots, x^n\}$ is a basis of $\mathcal{P}_n \Rightarrow$ dim. $\mathcal{P}_n = n + 1$

$$\Rightarrow \Phi_{\mathcal{B}}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = [\, a_0 \;\; a_1 \;\; a_2 \;\; \cdots \;\; a_n \,]^T.$$

Example: In $\mathcal{M}_{m\times n}$, let E_{ij} be the matrix whose (i,j)-entry is 1 and other entries are all 0, and let $S = \{E_{ij} \mid 1 \le i \le m,\ 1 \le j \le n\}$.

$\Rightarrow$ S is a basis of $\mathcal{M}_{m\times n}$. $\Rightarrow$ dim. $\mathcal{M}_{m\times n} = mn$.

For $m = n = 2$,

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and for all $A \in \mathcal{M}_{2\times 2}$,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Example: Define $U : \mathcal{M}_{m\times n} \to \mathcal{L}(\mathcal{R}^n, \mathcal{R}^m)$ by $U(A) = T_A$, the matrix transformation induced by A.

$\Rightarrow \forall\ A, B \in \mathcal{M}_{m\times n}$ and scalar c, $U(cA) = T_{cA} = cT_A = cU(A)$ and

$U(A + B) = T_{A+B} = T_A + T_B = U(A) + U(B)$

$\Rightarrow U$ is linear.

Also, U is an isomorphism (you show it).

$\Rightarrow$ dim. $\mathcal{L}(\mathcal{R}^n, \mathcal{R}^m) = mn$.

Homework Set for Section 7.3

- Section 7.3: Problems 1, 9, 19, 21, 26, 29, 39, 45, 50, 51, 54, 55, 60, 62, 63.