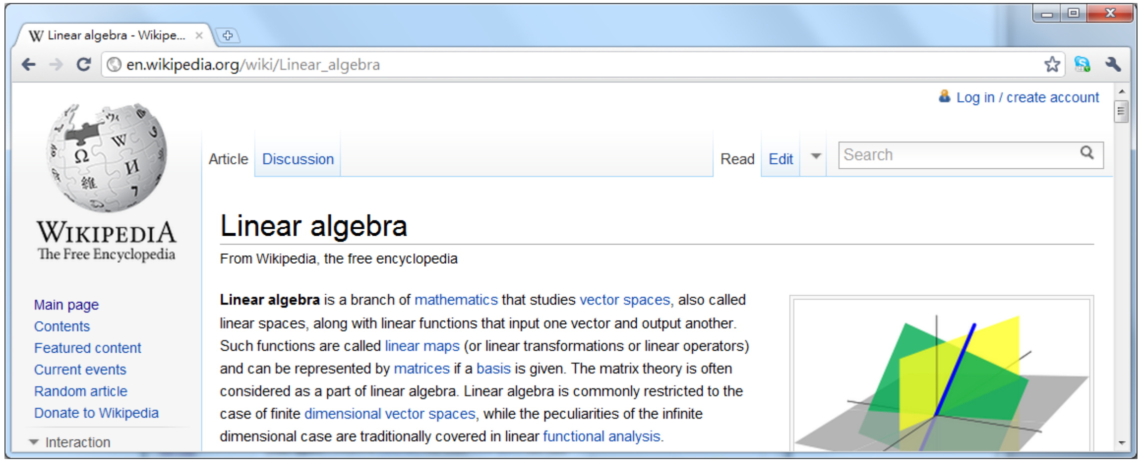
What is Linear Algebra?

- Wikipedia:
  - **Linear algebra** is the branch of **mathematics** concerning **vector spaces**, often finite or countably infinite **dimensional**, as well as **linear mappings** between such spaces. Such an investigation is initially motivated by a **system of linear equations** containing several unknowns. Such equations are naturally represented using the formalism of **matrices** and **vectors**.



CHAPTER 7

VECTOR SPACES

Section 7.1 Vector Spaces and Their Subspaces

Definition. (Vector Space)

A vector space V over a field $\mathcal{F}$ with two operations **vector addition** "+" and **scalar multiplication** "$\cdot$" defined so that

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$.
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
3. There is an element in V, denoted $\mathbf{0}$, such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for any $\mathbf{u}\in V$.
4. For any $\mathbf{u}\in V$, there exists an element, denoted $-\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
5. $1\cdot\mathbf{u}=\mathbf{u}$ for all $\mathbf{u}\in V$.
6. $(ab)\cdot\mathbf{u}=a\cdot(b\mathbf{u})$ for all $a,b\in\mathcal{F}$ and $\mathbf{u}\in V$.
7. $a\cdot(\mathbf{u}+\mathbf{v})=a\cdot\mathbf{u}+a\cdot\mathbf{v}$ for all $a\in\mathcal{F}$ and $\mathbf{u},\mathbf{v}\in V$.
8. $(a+b)\cdot\mathbf{u}=a\cdot\mathbf{u}+b\cdot\mathbf{u}$ for all $a,b\in\mathcal{F}$ and $\mathbf{u}\in V$.

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Four essential components in the definition of vector spaces

1. The set of vectors: V.
2. The field: $\mathcal{F}$. (along with "+" and "." operations thereof)
3. Addition operator "+": $V \times V \rightarrow V$
4. Scalar multiplication operation ".": $\mathcal{F} \times V \rightarrow V$

Proposition. ($\mathcal{R}^n$, $\mathcal{R}$, "+", ".") is a vector space where vector addition "+" and scalar multiplication "." were defined as in Chapter 1.

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Question. Are there other forms of vector spaces?

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Theorem 1.1 (Properties of Matrix Addition and Scalar Multiplication)

Let A, B, and C be $m\times n$ matrices, and let s and t be any scalars. Then

(a) $A+B=B+A$. (commutative law of matrix addition)
(b) $(A+B)+C=A+(B+C)$. (associative law of matrix addition)
(c) $A+O=A$.
(d) $A+(-A)=O$.
(e) $(st)A=s(tA)$.
(f) $s(A+B)=sA+sB$.
(g) $(s+t)A=sA+tA$.

Consider the case where $n=1$,
Then A, B, C are all vectors in $\mathcal{R}^m$.

Properties:

1. $\mathcal{R}^n$ is a vector space under the operations of sum and scalar multiplication defined in Chapter 1.
2. $\mathcal{M}_{m\times n}$ is a vector space under the operations of matrix addition and multiplication of a matrix by a scalar.
3. Any subspace of $\mathcal{R}^n$ is a vector space.

Definition. (Function Space)

Given a set S, let $\mathcal{F}(S) = \{f(\cdot)|f(t) \in \mathcal{R}, \forall t \in S\}$, and let there be two operations on $\mathcal{F}(S)$:

1. $\forall f, g \in \mathcal{F}(S)$, the sum $f + g$ satisfies $(f + g)(t) = f(t) + g(t), \forall t \in S$
2. $\forall f \in \mathcal{F}(S)$ and $a \in \mathcal{R}$, the scalar multiple af satisfies $(af)(t) = a(f(t)), \forall t \in S$.

Then $\mathcal{F}(S)$ is called a **function space**.

Theorem 7.1

$\mathcal{F}(S)$ is a vector space.

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$\mathcal{F}(S)$ is a vector space.

Proof Clearly, $f + g \in \mathcal{F}(S)$ and $af \in \mathcal{F}(S)$.

The Axioms 1 and 3 are verified below. You verify others.

Axiom 1: $(f + g)(t) = f(t) + g(t) = g(t) + f(t) = (g + f)(t)\ \forall\ t \in S$

$\Rightarrow f + g = g + f \quad \forall\ f, g \in \mathbb{F}(S).$

Axiom 3: define $\mathbf{0}(\cdot) \in \mathcal{F}(S)$ as $\mathbf{0}(t) = 0\ \forall\ t \in S$

$\Rightarrow (f + \mathbf{0})(t) = f(t) + \mathbf{0}(t) = f(t) + 0 = f(t)$

$\Rightarrow f + \mathbf{0} = f \quad \forall\ f \in \mathcal{F}(S).$

Example

For any given positive integers m and n, let $\mathcal{M}_{m\times n}$ denote the set of all $m \times n$ real matrices. Then $(\mathcal{M}_{m\times n}, \mathcal{R}, +, \cdot)$ is a vector space.

Definition

$$\mathcal{L}(\mathcal{R}^n, \mathcal{R}^m) = \{\text{all linear transformations from } \mathcal{R}^n \text{ to } \mathcal{R}^m\}$$

Property: $\mathcal{L}(\mathcal{R}^n, \mathcal{R}^m)$ is a vector space under the operations of addition of linear transformations and the product of a linear transformation by a scalar.

Proof You show it, noting that the "zero vector" is T_0.

Definition

$\mathcal{P} = \{\text{all polynomials in the variable } x\}$.

Property: $\mathscr{P}$ is a vector space with the usual definitions of polynomial addition and the product of a polynomial by a scalar.

Proof You show it, noting that the "zero vector" is $p(x) = 0$, the zero polynomial (whose degree is defined to be $-\infty$).

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Question: Why isn't there an axiom like $0\cdot\mathbf{v}=\mathbf{0}$ or $a\cdot\mathbf{0}=\mathbf{0}$

Theorem 7.2

Let V be a vector space. Then for any $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and any scalar $a \in \mathcal{F}$, the following are true.

(a) If $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{w}$. (right cancellation law)

(b) If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$. (left cancellation law)

(c) The zero vector $\mathbf{0}$ is unique, that is, it is the only vector in V that satisfies axiom 3.

(d) Each vector in V has exactly one additive inverse.

(e) $0 \cdot \mathbf{v} = \mathbf{0}$.

(f) $a \cdot \mathbf{0} = \mathbf{0}$.

(g) $(-1) \cdot \mathbf{v} = -\mathbf{v}$.

(h) $(-a) \cdot \mathbf{v} = a \cdot (-\mathbf{v}) = -(a \cdot \mathbf{v})$.

(a) $\underline{u} \overset{\text{axiom 3}}{=} \underline{u} + \underline{0} \overset{\text{axiom 4}}{=} \underline{u} + (\underline{v} + (-\underline{v})) \overset{\text{axiom 2}}{=} (\underline{u} + \underline{v}) + (-\underline{v}) = (\underline{w} + \underline{v}) + (-\underline{v})$

$\overset{\text{axiom 2}}{=} \underline{w} + (\underline{v} + (-\underline{v})) \overset{\text{axiom 4}}{=} \underline{w} + \underline{0} \overset{\text{axiom 3}}{=} \underline{w}$

(f) $a \cdot \underline{0} = \underline{w} \in V$ $\quad \underline{w} + \underline{w} = a \cdot \underline{0} + a \cdot \underline{0} \overset{\text{axiom 7}}{=} a \cdot (\underline{0} + \underline{0}) \overset{\text{axiom 3}}{=} a \cdot \underline{0} = \underline{w} = \underline{0} + \underline{w}$

Apply Thm 7.2(a) $\Rightarrow \underline{w} = \underline{0} \Rightarrow a \cdot \underline{0} = \underline{0}$

Proof (a)
$$\begin{aligned}\mathbf{u} &= \mathbf{u}+\mathbf{0} && \text{(by axiom 3)}\\ &= \mathbf{u}+(\mathbf{v}+(-\mathbf{v})) && \text{(by axiom 4)}\\ &= (\mathbf{u}+\mathbf{v})+(-\mathbf{v})) && \text{(by axiom 2)}\\ &= (\mathbf{w}+\mathbf{v})+(-\mathbf{v})) && \\ &= \mathbf{w}+(\mathbf{v}+(-\mathbf{v})) && \text{(by axiom 2)}\\ &= \mathbf{w}+\mathbf{0} && \text{(by axiom 4)}\\ &= \mathbf{w}. && \text{(by axiom 3)}\end{aligned}$$

(b) Similar to (a). You show it.

(c) Suppose $\mathbf{0}'$ and $\mathbf{0}$ are such that $\mathbf{u}+\mathbf{0}'=\mathbf{u}=\mathbf{u}+\mathbf{0}\ \forall \mathbf{u}\in V$.

$\Rightarrow \mathbf{0}'=\mathbf{0}$ by (b).

(d) Similar to (c). You show it.

(e)
$$\begin{aligned}0\mathbf{v}+0\mathbf{v} &= (0+0)\mathbf{v} && \text{(by axiom 8)}\\ &= 0\mathbf{v} && \text{(property of 0)}\\ &= \mathbf{0}+0\mathbf{v}. && \text{(by axioms 3 and 1)}\end{aligned}$$

$\Rightarrow 0\mathbf{v}=\mathbf{0}$ by (a).

(f) $\mathbf{0}+a\mathbf{0}=a\mathbf{0}=a(\mathbf{0}+\mathbf{0})=a\mathbf{0}+a\mathbf{0} \Rightarrow a\mathbf{0}=\mathbf{0}$ by (a).

(g) $\mathbf{v}+(\text{-}1)\mathbf{v}=(1)\mathbf{v}+(\text{-}1)\mathbf{v}=(1+(\text{-}1))\mathbf{v}=0\mathbf{v}=\mathbf{0}$

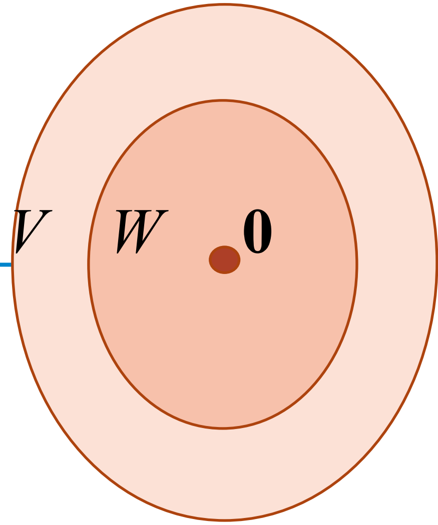
$\Rightarrow (\text{-}1)\mathbf{v}=\text{-}\mathbf{v}$ by (d).

(h) Similar to (g). You show it.

Definition. (Subspace)

A subset W of a vector space V is called a **subspace** of V if W satisfies the following three properties.

1. The zero vector of V is in W.
2. For all $\mathbf{u}, \mathbf{v}$ in W, $\mathbf{u} + \mathbf{v}$ is in W.
3. For all $\mathbf{u}$ in W and any scalar c, $c \cdot \mathbf{u}$ is in W.



Properties:

1. A subspace is itself a vector space.
2. V is the largest subspace of a vector space V.
3. The zero subspace $\{\mathbf{0}\}$ is a subspace of any vector space V.

Example: Given $S \neq \emptyset$ and $s_0 \in S$, $W = \{ f(\cdot) \mid f(s_0) = 0\} \subseteq \mathcal{F}(S)$ is a subspace of $\mathcal{F}(S)$ since for all $f(\cdot), g(\cdot) \in W$ and $a \in \mathcal{R}$,

$$(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0, \text{ and}$$

$$(af)(s_0) = af(s_0) = a \cdot 0 = 0.$$

$M_{n\times n} = R^{n\times n}$

Example:

Define a trace of an n x n matrix A as

$$\text{trace } A = a_{11} + a_{22} + \ldots + a_{nn}.$$

Then,

$W = \{ A \mid \text{trace } A = 0\} \subseteq \mathcal{R}^{n\times n}$ is a subspace of $\mathcal{R}^{n\times n}$ since for all $A, B \in W$ and $c \in \mathcal{R}$,

$$\text{trace}(A + B) = \text{trace } A + \text{trace } B = 0 + 0 = 0, \text{ and}$$

$$\text{trace } (cA) = c \cdot \text{trace } A = c\cdot 0 = 0.$$

Example: $C(\mathcal{R})$ = {all continuous functions mapping from $\mathcal{R}$ to $\mathcal{R}$} is a subspace of $\mathcal{F}(\mathcal{R})$, since the sum of two continuous functions is a continuous function, and any scalar multiple of a continuous function is a continuous function.

$\Rightarrow$ $C(\mathcal{R})$ is a vector space.

Example: $\mathscr{P}_n$ = {all polynomials in the variable x with degree $\leq n$}
is a subspace of $\mathscr{P}$.
$\Rightarrow$ $\mathscr{P}_n$ is a vector space.

Example: Let $\mathscr{P}'_n$ = {all polynomials in the variable x with degree $= n$}
Is $\mathscr{P}'_n$ a subspace of $\mathscr{P}$?
Is $\mathscr{P}'_n$ a vector space.

$(x^2+1)+(-x^2+1)=2$

Definition. (Linear Combination)

A vector $\mathbf{v}$ is a **linear combination** of the vectors of a (possibly **infinite**) subset S of a vector space V if there exist (**finite number of**) vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ in S and scalars $c_1, c_2, \cdots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The scalars are called the **coefficients** of the linear combination.

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Example: A linear combination in the vector space $\mathcal{R}^{2\times 2}$

$$\underline{\begin{bmatrix} -1 & 8 \\ 2 & -2 \end{bmatrix}} = 2\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} + (-1)\begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} + (1)\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right\} \subseteq R^{2\times 2}$$

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Example: A linear combination in the vector space $\mathscr{P}$

$$f(x) = 2 + 3x - x^2 = (2)1 + (3)x + (-1)x^2 + (0)x^3$$

$S_3 = \{1, x, x^2, x^3\}$

$\Rightarrow \mathscr{P}_3 =$ the set of all linear combinations of the vectors in S_3.

$S = \{1, x, x^2, \cdots, x^n, \cdots\}$

$\Rightarrow$ any polynomial is a linear combination of the vectors in S.

$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in P$

$1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = e^x \notin P$

Definition. (Span)

The **span** of a nonempty subset S of a vector space V is the set of all linear combinations of vectors in S. This set is denoted Span S:

Example: $\text{Span}\ \{1, x, x^2, x^3\} = \mathcal{P}_3$ $\quad$ $\text{Span}\ \{1, x, x^2, \ldots, x^n, \ldots\} = \mathcal{P}$

Example: For all $A \in$ Span S, with $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \subset \mathcal{R}^{2\times 2}$,

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

$\Rightarrow$ Span $S = \{A \mid \text{trace } A = 0, A \in \mathcal{R}^{2\times 2}\}$, a subspace of $\mathcal{R}^{2\times 2}$.

Question:
Let V be a vector space and W be the span of a subset S.
Is W a subspace of V?

Theorem 7.3
The span of a nonempty subset of a vector space V is a subspace of V.

Proof Extend the proof of Theorem 4.1 in Section 4.1.

(1) $\underline{0} \in W$

(2) $\underline{u}+\underline{v} \in W \quad \forall\, \underline{u}, \underline{v} \in W$

(3) $c \cdot \underline{u} \in W \quad \forall\, \underline{u} \in W,\ c \in R$

Example: the space of trigonometric polynomials

$$\mathcal{T}[0,2\pi] = \text{Span } \{1, \cos t, \sin t, \cos 2t, \sin 2t, \cdots, \cos nt, \sin nt, \cdots\}$$

is a subspace of $\mathcal{F}([0, 2\pi])$,

Homework Set for Section 7.1

- Section 7.1: Problems 28, 30, 32, 55-59, 79-82, 91, 92.