Section 6.2 Orthogonal Vectors

Definition

A subset of $\mathcal{R}^n$ is called an **orthogonal set** if every pair of distinct vectors in the set is orthogonal.

Example: an orthogonal set

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$$

By definition, a set with only one vector is an orthogonal set.

Question: Is an orthogonal set also linearly independent?

Theorem 6.5

Any orthogonal set of nonzero vectors is linearly independent.

Proof Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathcal{R}^n$ be an orthogonal set and $\mathbf{v}_i \neq \mathbf{0}$ for $i = 1, 2, \ldots, k$.

If $c_1, c_2, \ldots, c_k$ make $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, then for $i = 1, 2, \ldots, k$, we have

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If $c_1, c_2, \ldots, c_k$ make $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, then for $i = 1, 2, \ldots, k$, we have

$$\begin{aligned}
0 &= \mathbf{0} \cdot \mathbf{v}_i \\
&= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_i\mathbf{v}_i + \cdots + c_k\mathbf{v}_k) \cdot \mathbf{v}_i \\
&= c_1\mathbf{v}_1 \cdot \mathbf{v}_i + c_2\mathbf{v}_2 \cdot \mathbf{v}_i + \cdots + c_i\mathbf{v}_i \cdot \mathbf{v}_i + \cdots + c_k\mathbf{v}_k \cdot \mathbf{v}_i \\
&= c_i(\mathbf{v}_i \cdot \mathbf{v}_i) \\
&= c_i\underbrace{||\mathbf{v}_i||^2}_{\neq 0}, \text{ i.e., } c_i = 0.
\end{aligned}$$

Definition

A basis that is an **orthogonal set** is called an **orthogonal basis**.

Example: The standard basis $\mathcal{E}$ of $\mathcal{R}^n$ is an orthogonal basis.

Question

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be an **orthogonal basis** for a subspace V of $\mathcal{R}^n$, and let $\mathbf{u}$ be a vector in V. How do we express $\mathbf{u}$ as a linear combination of vectors in S?

Representation of a Vector in Terms of an Orthogonal Basis

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be an orthogonal basis for a subspace V of $\mathcal{R}^n$, and let $\mathbf{u}$ be a vector in V. Then

$$\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\mathbf{u} \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \cdots + \frac{\mathbf{u} \cdot \mathbf{v}_k}{||\mathbf{v}_k||^2}\mathbf{v}_k$$

Proof

Representation of a Vector in Terms of an Orthogonal Basis

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be an orthogonal basis for a subspace V of $\mathcal{R}^n$, and let $\mathbf{u}$ be a vector in V. Then

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Proof $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \Rightarrow$

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v}_i &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_i\mathbf{v}_i + \cdots + c_k\mathbf{v}_k) \cdot \mathbf{v}_i \\
&= c_1\mathbf{v}_1 \cdot \mathbf{v}_i + c_2\mathbf{v}_2 \cdot \mathbf{v}_i + \cdots + c_i\mathbf{v}_i \cdot \mathbf{v}_i + \cdots + c_k\mathbf{v}_k \cdot \mathbf{v}_i \\
&= c_i(\mathbf{v}_i \cdot \mathbf{v}_i) \\
&= c_i\|\mathbf{v}_i\|^2 \Rightarrow c_i = \frac{\mathbf{u} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2}.
\end{aligned}$$

Example: $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathcal{R}^3$ with

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$\text{Let } \mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3. \quad \Rightarrow$$

$$c_1 = \frac{\mathbf{u}\cdot\mathbf{v}_1}{||\mathbf{v}_1||^2} = \frac{10}{14}, c_2 = \frac{\mathbf{u}\cdot\mathbf{v}_2}{||\mathbf{v}_2||^2} = \frac{4}{3}, c_3 = \frac{\mathbf{u}\cdot\mathbf{v}_3}{||\mathbf{v}_3||^2} = \frac{8}{42}.$$

Question

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be a basis for a subspace V of $\mathcal{R}^n$, and let $\mathbf{u}$ be a vector in V. How do we find, systematically, an **orthogonal basis** for V from the values of vectors in S?

Example: $\boldsymbol{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where $\mathbf{u}_1 = [\ 1\ \ 1\ \ 1\ \ 1\ ]^T$, $\mathbf{u}_2 = [\ 2\ \ 1\ \ 0\ \ 1\ ]^T$, and $\mathbf{u}_3 = [\ 1\ \ 1\ \ 2\ \ 1\ ]^T$ are L.I. vectors. Let's find an orthogonal basis for Span $\boldsymbol{S}$.

Theorem 6.5 (Gram-Schmidt Process)

Let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ be a basis for a subspace W of $\mathcal{R}^n$. Define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1, \\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2, \\
&\vdots \\
\mathbf{v}_k &= \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_k \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 - \cdots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{||\mathbf{v}_{k-1}||^2}\mathbf{v}_{k-1}.
\end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_i\}$ is an orthogonal set of nonzero vectors such that

$$\text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_i\} = \text{Span}\ \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_i\}$$

for each i. So $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is an **orthogonal basis** for W.

Theorem 6.5 (Gram-Schmidt Process)

Let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ be a basis for a subspace W of $\mathcal{R}^n$. Define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1, \\
&\vdots \\
\mathbf{v}_k &= \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_k \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 - \cdots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{||\mathbf{v}_{k-1}||^2}\mathbf{v}_{k-1}.
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for each i. So $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is an **orthogonal basis** for W.

Proof By induction on k.

The theorem obviously holds for $k = 1$.

Assume the theorem holds for $k \geq 1$, and consider the case for $k + 1$.

We have

1. In the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are nonzero orthogonal vectors, and $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.
2. $\mathbf{v}_{k+1} \cdot \mathbf{v}_i = 0$ for $i = 1, 2, \ldots, k$, since

$$\mathbf{v}_{k+1}\cdot\mathbf{v}_i = \mathbf{u}_{k+1}\cdot\mathbf{v}_i - \frac{\mathbf{u}_{k+1}\cdot\mathbf{v}_1}{\|\mathbf{v}_1\|^2}\underbrace{\mathbf{v}_1\cdot\mathbf{v}_i}_{0} - \cdots - \frac{\mathbf{u}_{k+1}\cdot\mathbf{v}_{i-1}}{\|\mathbf{v}_{i-1}\|^2}\underbrace{\mathbf{v}_{i-1}\cdot\mathbf{v}_i}_{0}$$

$$- \frac{\mathbf{u}_{k+1}\cdot\mathbf{v}_i}{\cancel{\|\mathbf{v}_i\|^2}}\cancel{\mathbf{v}_i\cdot\mathbf{v}_i} - \frac{\mathbf{u}_{k+1}\cdot\mathbf{v}_{i+1}}{\|\mathbf{v}_{i+1}\|^2}\underbrace{\mathbf{v}_{i+1}\cdot\mathbf{v}_i}_{0} - \cdots - \frac{\mathbf{u}_{k+1}\cdot\mathbf{v}_k}{\|\mathbf{v}_k\|^2}\underbrace{\mathbf{v}_k\cdot\mathbf{v}_i}_{0}.$$

3. $\mathbf{v}_{k+1} \neq \mathbf{0}$, since otherwise

$$\mathbf{0} = \mathbf{u}_{k+1} - \frac{\mathbf{u}_{k+1}\cdot\mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{u}_{k+1}\cdot\mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \cdots - \frac{\mathbf{u}_{k+1}\cdot\mathbf{v}_k}{\|\mathbf{v}_k\|^2}\mathbf{v}_k,$$

which implies $\mathbf{u}_{k+1} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.

4. $\mathbf{v}_{k+1} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_{k+1}\} \subseteq \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$
$\Rightarrow \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\} \subseteq \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$

5. $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ is L.I. $\Rightarrow$ dim.Span$\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\} = k+1$,
$\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ is orthogonal $\Rightarrow$ $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ is L.I.
$\Rightarrow$ dim.Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\} = k+1$,
$\Rightarrow \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\} = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$

Example: $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where $\mathbf{u}_1 = [\ 1\ \ 1\ \ 1\ \ 1\ ]^T$, $\mathbf{u}_2 = [\ 2\ \ 1\ \ 0\ \ 1\ ]^T$, and $\mathbf{u}_3 = [\ 1\ \ 1\ \ 2\ \ 1\ ]^T$ are L.I. vectors. Let

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix},$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{u}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{u}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} - \frac{5}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(-1)}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Then $\mathcal{S}' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for Span$\mathcal{S}$.

To multiply nonzero scalars on any vectors of $\mathcal{S}'$ will not affect its being an orthogonal basis. For example, $\mathcal{S}'' = \{\mathbf{v}_1, \mathbf{v}_2, 4\mathbf{v}_3\}$ is still an orthogonal basis.

Definition

A vector that has **norm** equal to unity is called a **unit vector**.

Property: For any nonzero vector $\mathbf{v}$, $(1/\|\mathbf{v}\|)\mathbf{v}$ is a unit vector.

Definition

An orthogonal basis consisting of **unit vectors** is called an **orthonormal basis**.

Property: Every nonzero subspace has an orthonormal basis.

Example: $\mathcal{S}' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for Span $\mathcal{S}$, where $\mathcal{S} = \{[\,1\;\;1\;\;1\;\;1\,]^T, [\,2\;\;1\;\;0\;\;1\,]^T, [\,1\;\;1\;\;2\;\;1\,]^T\}$.

Since $||\mathbf{v}_1|| = 2$, $||\mathbf{v}_2|| = \sqrt{2}$, $||\mathbf{v}_3|| = \frac{1}{2}$, then set

$$\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \left\{\frac{1}{2}\mathbf{v}_1, \frac{1}{\sqrt{2}}\mathbf{v}_2, 2\mathbf{v}_3\right\}$$

s an orthonormal basis for Span S.

Question

Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$ be an **orthonormal basis** for a subspace W of $\mathcal{R}^n$, and let $\mathbf{u}$ be a vector in W. How do we express $\mathbf{u}$ as a linear combination of vectors in S?

Representation of a Vector in Terms of an Orthonormal Basis

Let $\mathcal{S} = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$ be an orthonormal basis for a subspace V of $\mathcal{R}^n$, and let $\mathbf{u}$ be a vector in V. Then

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2 + \cdots + (\mathbf{u} \cdot \mathbf{w}_k)\mathbf{w}_k.$$

Proof

Representation of a Vector in Terms of an Orthonormal Basis

Let $\mathcal{S} = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$ be an orthonormal basis for a subspace V of $\mathcal{R}^n$, and let $\mathbf{u}$ be a vector in V. Then

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2 + \cdots + (\mathbf{u} \cdot \mathbf{w}_k)\mathbf{w}_k.$$

Proof If $\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k$, then $c_i = \mathbf{u} \cdot \mathbf{w}_i / \|\mathbf{w}_i\|^2 = \mathbf{u} \cdot \mathbf{w}_i$.

Example: In the above example, $\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthonormal basis for Span$\mathcal{S}$, where $\mathcal{S} = \{[\,1\ \ 1\ \ 1\ \ 1\,]^T, [\,2\ \ 1\ \ 0\ \ 1\,]^T, [\,1\ \ 1\ \ 2\ \ 1\,]^T\}$. For $\mathbf{u} = [\,2\ \ 3\ \ 5\ \ 3\,]^T \in \text{Span}\,\mathbb{S}$,

$$\begin{aligned} \mathbf{u} &= (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2 + (\mathbf{u} \cdot \mathbf{w}_3)\mathbf{w}_3 \\ &= \frac{13}{2}\mathbf{w}_1 + \left(\frac{-3}{\sqrt{2}}\right)\mathbf{w}_2 + \frac{1}{2}\mathbf{w}_3. \end{aligned}$$

Homework Set for Section 6.2

- Section 6.2: Problems 5, 8, 9, 14, 16, 53, 54, 55