Chapter 6: Orthogonality
Section 6.1: The Geometry of Vectors

Linear Algebra

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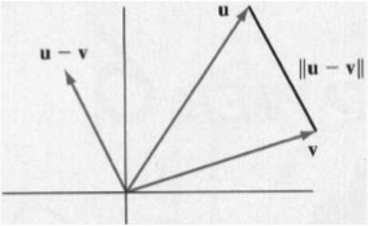
Norm, length, and distance

Definition

Let $\mathbf{v}$ be any vector in $\mathcal{R}^n$. Then norm (length) of $\mathbf{v}$, denoted $||\mathbf{v}||$, is defined by

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$ is defined by $||\mathbf{u} - \mathbf{v}||$.



Norm and distance – an example

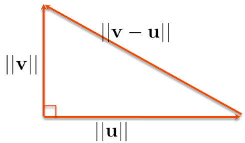
Example:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}.$$

$$\Rightarrow ||\mathbf{u}|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14},$$

$$||\mathbf{u} - \mathbf{v}|| = \sqrt{(1-2)^2 + (2-(-3))^2 + (3-0)^2} = \sqrt{35}.$$

Perpendicularity in $\mathcal{R}^2$: the Pythagorean Theorem



geometric condition for perpendicularity in $\mathcal{R}^2$

$$||\mathbf{v} - \mathbf{u}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

algebraic condition for perpendicularity in $\mathcal{R}^2$

$$u_1 v_1 + u_2 v_2 = 0$$

It can be shown that these two conditions are equivalent.

Geometric and algebraic conditions

Proposition

$||\mathbf{v}-\mathbf{u}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ is equivalent to $u_1v_1 + u_2v_2 = 0$.

Geometric and algebraic conditions

Proposition

$||\mathbf{v}-\mathbf{u}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ is equivalent to $u_1 v_1 + u_2 v_2 = 0$.

Proof

$$\begin{aligned}
||\mathbf{v}-\mathbf{u}||^2 &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \\
(v_1-u_1)^2 + (v_2-u_2)^2 &= u_1^2 + u_2^2 + v_1^2 + v_2^2 \\
v_1^2 - 2u_1 v_1 + u_1^2 + v_2^2 - 2u_2 v_2 + v_2^2 &= u_1^2 + u_2^2 + v_1^2 + v_2^2 \\
-2u_1 v_1 - 2u_2 v_2 &= 0 \\
u_1 v_1 + u_2 v_2 &= 0.
\end{aligned}$$

Dot product

Definition

Let **u** and **v** be vectors in $\mathcal{R}^n$. The dot product of **u** and **v** is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

We say that **u** and **v** are orthogonal (perpendicular) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Dot product

Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{R}^n$. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is defined by

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We say that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Properties

- $\mathbf{0}$ is orthogonal to every vector in $\mathcal{R}^n$.
- For $\mathbf{u}, \mathbf{v} \in \mathcal{R}^3$, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Dot product

Properties

- The dot product can also be represented by the matrix product

$$
\begin{aligned}
\mathbf{u}^T\mathbf{v} &= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \\
&= \mathbf{u} \cdot \mathbf{v}.
\end{aligned}
$$

Properties of dot products

Theorem 6.1

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{R}^n$ and c be a scalar in $\mathcal{R}$.
(a) $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$.
(b) $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
(c) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
(d) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
(e) $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$.
(f) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.
(g) $||c\mathbf{u}|| = |c|||\mathbf{u}||$.

Proof

All results may be proven easily using the definition.

Properties of dot products

Corollary

- For $c \in \mathcal{R}$, $\mathbf{u}, \mathbf{v} \in \mathcal{R}^n$, $c\mathbf{u} \cdot \mathbf{v}$ has a clear meaning: $(c\mathbf{u}) \cdot \mathbf{v}$ or $c(\mathbf{u} \cdot \mathbf{v})$.
- For $c_1, c_2, \cdots, c_p \in \mathcal{R}, \mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p \in \mathcal{R}^n$,

$$\mathbf{u} \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) = c_1\mathbf{u} \cdot \mathbf{v}_1 + c_2\mathbf{u} \cdot \mathbf{v}_2 + \cdots + c_p\mathbf{u} \cdot \mathbf{v}_p$$

and

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) \cdot \mathbf{u} = c_1\mathbf{v}_1 \cdot \mathbf{u} + c_2\mathbf{v}_2 \cdot \mathbf{u} + \cdots + c_p\mathbf{v}_p \cdot \mathbf{u}.$$

Example

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Show that

$$||2\mathbf{u} + 3\mathbf{v}||^2 = 4||\mathbf{u}||^2 + 12(\mathbf{u} \cdot \mathbf{v}) + 9||\mathbf{v}||^2$$

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Show that

$$||2\mathbf{u} + 3\mathbf{v}||^2 = 4||\mathbf{u}||^2 + 12(\mathbf{u} \cdot \mathbf{v}) + 9||\mathbf{v}||^2$$

$$\begin{aligned}
||2\mathbf{u} + 3\mathbf{v}||^2 &= (2\mathbf{u} + 3\mathbf{v}) \cdot (2\mathbf{u} + 3\mathbf{v}) \\
&= (2\mathbf{u}) \cdot (2\mathbf{u} + 3\mathbf{v}) + (3\mathbf{v}) \cdot (2\mathbf{u} + 3\mathbf{v}) \\
&= (2\mathbf{u}) \cdot (2\mathbf{u}) + (2\mathbf{u}) \cdot (3\mathbf{v}) + (3\mathbf{v}) \cdot (2\mathbf{u}) + (3\mathbf{v}) \cdot (3\mathbf{v}) \\
&= 4(\mathbf{u} \cdot \mathbf{u}) + 6(\mathbf{u} \cdot \mathbf{v}) + 6(\mathbf{v} \cdot \mathbf{u}) + 9(\mathbf{v} \cdot \mathbf{v}) \\
&= 4||\mathbf{u}||^2 + 6(\mathbf{u} \cdot \mathbf{v}) + 6(\mathbf{v} \cdot \mathbf{u}) + 9||\mathbf{v}||^2 \\
&= 4||\mathbf{u}||^2 + 12(\mathbf{u} \cdot \mathbf{v}) + 9||\mathbf{v}||^2
\end{aligned}$$

Pythagoream Theorem in $\mathcal{R}^n$

Theorem 6.2

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{R}^n$. Then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.$$

Proof.

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 &= ||\mathbf{u}||^2 + 2\mathbf{u}\cdot\mathbf{v} + ||\mathbf{v}||^2 \\ &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \end{aligned}$$

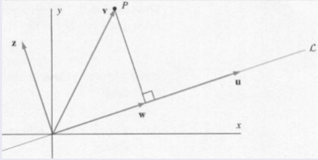
if and only if

$$\mathbf{u}\cdot\mathbf{v} = 0.$$

$\square$

Orthogonal projection of a vector onto a line

orthogonal projection



$\mathbf{v}$: any vector in $\mathcal{R}^2$.
$\mathbf{u}$: any nonzero vector on $\mathcal{L}$.
$\mathbf{w}$: orthogonal projection of $\mathbf{v}$ onto $\mathcal{L}$, $\mathbf{w} = c\mathbf{u}$.
$\mathbf{z}$: $\mathbf{v} - \mathbf{w}$.

$$0 = \mathbf{z} \cdot \mathbf{u} = (\mathbf{v} - \mathbf{w}) \cdot \mathbf{u} = (\mathbf{v} - c\mathbf{u}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} - c\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} - c||\mathbf{u}||^2$$

$$\Rightarrow c = \frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2} \text{ and } \mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2}\mathbf{u}.$$

Orthogonal projection of a vector onto a line

orthogonal projection

Distance from P (tip of $\mathbf{v}$) to $\mathcal{L}$:

$$\left|\left|\mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2}\mathbf{u}\right|\right|$$

Orthogonal projection of a vector onto a line

example

$\mathcal{L} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{R}^2 \,\middle|\, y = (1/2)x \right\}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Then

$$\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2}\mathbf{u} = \frac{9}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and the distance from the tip of $\mathbf{v}$ to $\mathcal{L}$ is

$$\left|\left| \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \frac{9}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right|\right| = \frac{1}{5}\left|\left| \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right|\right| = \frac{2}{5}\sqrt{5}$$

Cauchy-Schwarz Inequality

Theorem 6.3

For any vectors **u** and **v** in $\mathcal{R}^n$, we have

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$$

Cauchy-Schwarz Inequality

Theorem 6.3

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$, we have

$$|\mathbf{u}\cdot\mathbf{v}| \le ||\mathbf{u}||\cdot||\mathbf{v}||$$

Proof.

This inequality holds when $\mathbf{u}=\mathbf{0}$. Assume $\mathbf{u}\neq\mathbf{0}$. Then

$$\begin{aligned}
||\mathbf{v}||^2 = \left|\left|\frac{\mathbf{v}\cdot\mathbf{u}}{||\mathbf{u}||^2}\mathbf{u}+\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{u}}{||\mathbf{u}||^2}\mathbf{u}\right|\right|^2 &= \left|\left|\frac{\mathbf{v}\cdot\mathbf{u}}{||\mathbf{u}||^2}\mathbf{u}\right|\right|^2+\left|\left|\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{u}}{||\mathbf{u}||^2}\mathbf{u}\right|\right|^2 \\
&\ge \left|\left|\frac{\mathbf{v}\cdot\mathbf{u}}{||\mathbf{u}||^2}\mathbf{u}\right|\right|^2 = \frac{|\mathbf{v}\cdot\mathbf{u}|^2}{||\mathbf{u}||^2}.
\end{aligned}$$

Note: $\left(\frac{\mathbf{v}\cdot\mathbf{u}}{||\mathbf{u}||^2}\cdot\mathbf{u}\right)\left(\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{u}}{||\mathbf{u}||^2}\mathbf{u}\right)=\frac{(\mathbf{v}\cdot\mathbf{u})^2}{||\mathbf{u}||^2}-\frac{(\mathbf{v}\cdot\mathbf{u})^2}{||\mathbf{u}||^4}\mathbf{u}\cdot\mathbf{u}=0.$ □

Cauchy-Schwarz Inequality

Example

For any real numbers a_1, a_2, a_3, b_1, b_2, and b_3,

$$|a_1 b_1 + a_2 b_2 + a_3 b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}.$$

Triangle Inequality

Theorem 6.4

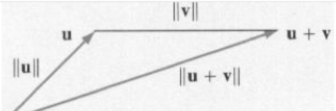
For any vectors **u** and **v** in $\mathcal{R}^n$, we have

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||.$$

Proof.

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 &= ||\mathbf{u}||^2 + 2\mathbf{u}\cdot\mathbf{v} + ||\mathbf{v}||^2 \\ &\leq ||\mathbf{u}||^2 + 2||\mathbf{u}||\cdot||\mathbf{v}|| + ||\mathbf{v}||^2 \\ &= (||\mathbf{u}|| + ||\mathbf{v}||)^2 \end{aligned}$$

$\square$



Complex dot product

For vectors in $\mathcal{C}^n$, the definition of dot products, as well as those of norm, distance, and orthogonality, needs to be modified a little bit.

Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{C}^n$. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1^* v_1 + u_2^* v_2 + \cdots + u_n^* v_n.$$

We say that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Complex dot product

Definition

Let $\mathbf{v}$ be any vector in $\mathcal{C}^n$. Then norm (length) of $\mathbf{v}$, denoted $||\mathbf{v}||$, is defined by

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{C}^n$ is defined by $||\mathbf{u} - \mathbf{v}||$.

Complex dot product

Theorem 6.1'

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{C}^n$ and c be a scalar in $\mathcal{C}$.

(a) $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$.

(b) $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

(c) $\mathbf{u} \cdot \mathbf{v} = (\mathbf{v} \cdot \mathbf{u})^*$.

(d) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

(e) $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$.

(f) $(c\mathbf{u}) \cdot \mathbf{v} = c^*(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c^*\mathbf{v})$.

(g) $||c\mathbf{u}|| = |c|||\mathbf{u}||$.

Homework set for Section 6.1

Section 6.1

Problems 7, 15, 81-89, 92, 95