Section 5.2 The Characteristic Polynomial

Question: Given an n x n matrix A, how do we find its eigenvalues?

Idea: Suppose c is an eigenvalue of A, then what is the determinant of A-cI?

Property

The eigenvalues of a square matrix A are the values of t that satisfy

$$\det(A - tI_n).$$

Proof $\exists$ $\mathbf{v} \neq \mathbf{0}$ such that $(A - tI_n)\mathbf{v} = \mathbf{0} \Leftrightarrow \det(A - tI_n) = 0$.

Section 5.2 The Characteristic Polynomial

Definition

$A \in \mathcal{R}^{n\times n}$, $\det(A - tI_n)$: **characteristic polynomial** of A; $\det(A - tI_n) = 0$: **characteristic equation** of A.

Example:

$$A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix} \quad \Longrightarrow \quad A - tI_2 = \begin{bmatrix} -4-t & -3 \\ 3 & 6-t \end{bmatrix} \quad \Longrightarrow$$

$$\det(A - tI_2) = (-4-t)(6-t) - (-3)\cdot 3 = (t+3)(t-5). \quad \Longrightarrow$$

So, eigenvalues of A can be -3 or 5.

Question: What can we find eigenvectors corresponding to -3?

What can we find eigenvectors corresponding to 5?

Example:

$$A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix} \quad \Rightarrow \quad A - tI_2 = \begin{bmatrix} -4-t & -3 \\ 3 & 6-t \end{bmatrix} \quad \Rightarrow$$

$$\det(A - tI_2) = (-4-t)(6-t) - (-3)\cdot 3 = (t+3)(t-5). \quad \Rightarrow$$

By solving $(A+3I_2)\mathbf{x} = 0$, we get a basis $\{[-3 \;\; 1]^T\}$ of the eigenspace of A corresponding to the eigenvalue -3.

By solving $(A-5I_2)\mathbf{x} = 0$, we get a basis $\{[-1 \;\; 3]^T\}$ of the eigenspace of A corresponding to the eigenvalue 5.

Question: Is the characteristic polynomial of A equal to its reduced row echelon form?

Question: What is the order of the characteristic polynomial of an $n\times n$ matrix A?

Properties:

1. In general, a matrix and its reduced row echelon form have different characteristic polynomials. Therefore, elementary operations are not useful in finding the characteristic polynomial of a matrix.
2. The characteristic polynomial of an $n \times n$ matrix is indeed a polynomial with degree n, as can be deduced from the complete determinant expansion of $\det(A - tI_n)$.
3. The eigenvalues of an upper triangular matrix are its diagonal entries.

Question: Now that **elementary operations** do not preserve **characteristic polynomials**, is there any other "operations" that do?

* Property

Similar matrices have the same characteristic polynomials.

Proof

Suppose A and B are $n \times n$ matrices that are similar. Assume $B = P^{-1}AP$ where P is invertible.

$\det(B - tI_n) =$

$= \det(A - tI_n)$

* Property

Similar matrices have the same characteristic polynomials.

Proof

$$\begin{aligned}
\det(B - tI_n) &= \det(P^{-1}AP - P^{-1}(tI_n)P) \\
&= \det(P^{-1}(A - tI_n)P) \\
&= (\det P^{-1})[\det(A - tI_n)](\det P) \\
&= \left(\frac{1}{\det P}\right)[\det(A - tI_n)](\det P) \\
&= \det(A - tI_n).
\end{aligned}$$

Definition

T: linear operator on $\mathcal{R}^n$ with the standard matrix A.

$\det(A - tI_n)$: **characteristic polynomial of T;**

$\det(A - tI_n) = 0$: **characteristic equation of T;**

Example: T : linear operator on $\mathcal{R}^2$ that rotates a vector by 90°.

$\Rightarrow$ standard matrix is the 90°-rotation matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow$ characteristic polynomial of T is

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - tI_2\right) = \det\begin{bmatrix} -t & -1 \\ 1 & -t \end{bmatrix} = t^2 + 1$$

Think: Consider an n x n matrix A.

How many eigenvalues does it have?

Can we "predict" the number of eigenvalues of A from its characteristic polynomial?

Fact: An n x n matrix A can have less than n eigenvalues due to the following reasons.

1) Complex (not real) solutions of its characteristic polynomial.
2) Multiplicity of a real eigenvalue.

Definition

If λ is an eigenvalue of an $n \times n$ matrix M, then the **largest positive integer** k such that $(t - \lambda)^k$ is a factor of the characteristic polynomial of M (i.e., $\det(M - tI_n)$), is called the **(algebraic) multiplicity** of λ.

Example: Find the characteristic polynomials of

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \text{ and } \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

The characteristic polynomials of

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \text{ and } \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

are $-(t + 1)^2(t - 3)$ and $-(t + 1)(t - 3)^2$, respectively, i.e.,

A: eigenvalue -1 3; multiplicity 2 1

B: eigenvalue -1 3; multiplicity 1 2

Then, find the eigenspaces corresponding to each of these eigenvalues.

The characteristic polynomials of

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \text{ and } \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

are $-(t + 1)^2(t - 3)$ and $-(t + 1)(t - 3)^2$, respectively, i.e.,

A: $\begin{cases} \text{eigenvalue} & -1 & 3 \\ \text{multiplicity} & 2 & 1 \end{cases}$ $\quad B$: $\begin{cases} \text{eigenvalue} & -1 & 3 \\ \text{multiplicity} & 1 & 2 \end{cases}$

Then, find the eigenspaces corresponding to each of these eigenvalues.

The characteristic polynomials of

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \text{ and } \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

are $-(t+1)^2(t-3)$ and $-(t+1)(t-3)^2$, respectively, i.e.,

$$A:\begin{cases} \text{eigenvalue} & -1 \quad 3 \\ \text{multiplicity} & 2 \quad 1 \end{cases} \qquad B:\begin{cases} \text{eigenvalue} & -1 \quad 3 \\ \text{multiplicity} & 1 \quad 2 \end{cases}$$

For A, the eigenspace corresponding to the eigenvalue 3 has a basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow \text{dimension} = 1 = \text{multiplicity of the eigenvalue } 3.$$

For B, the eigenspace corresponding to the eigenvalue 3 has a basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow \text{dimension} = 2 = \text{multiplicity of the eigenvalue } 3.$$

Facts & Questions:

1) The multiplicity of an eigenvalue λ of A is at least 1.
2) When the multiplicity of λ is 1, then must the dimension of its eigenspace be 1?
3) When the multiplicity of λ is k (where $k > 1$), then must the dimension of its eigenspace be k?

Example: Find The characteristic polynomial of

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Example: The characteristic polynomial of

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

is $-(t+1)(t-3)^2$, i.e.,

$$C: \begin{cases} \text{eigenvalue} & -1 & 3 \\ \text{multiplicity} & 1 & 2 \end{cases}$$

Now, find the eigenspace corresponding to each of these eigenvalues.

Example: The characteristic polynomial of

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

is $-(t+1)(t-3)^2$, i.e.,

$$C: \begin{cases} \text{eigenvalue} & -1 \quad 3 \\ \text{multiplicity} & 1 \quad 2 \end{cases}$$

For C, the eigenspace corresponding to the eigenvalue 3 has a basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \Rightarrow \text{dimension} = 1 < \text{multiplicity of the eigenvalue } 3.$$

Definition

The dimension of the eigenspace corresponding to the eigenvalue λ is called the **geometric multiplicity** of λ.

Theorem 5.1

Let λ be an eigenvalue of a matrix A. The dimension of the eigenspace of A corresponding to λ is less than or equal to the multiplicity of λ.

Theorem 5.1

Let λ be an eigenvalue of a matrix A. The dimension of the eigenspace of A corresponding to λ is less than or equal to the multiplicity of λ.

****Proof*** Suppose $A \in \mathcal{R}^{n\times n}$, the geometric multiplicity of λ is k, and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis of the eigenspace corresponding to λ.

$\Rightarrow$ can extend $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ to a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\mathcal{R}^n$.

Consider the linear operator T_A on $\mathcal{R}^n$. For $1 \leq j \leq k$,

$$[T_A(\mathbf{v}_j)]_{\mathcal{B}} = [A\mathbf{v}_j]_{\mathcal{B}} = [\lambda\mathbf{v}_j]_{\mathcal{B}} = \lambda[\mathbf{v}_j]_{\mathcal{B}} = \lambda\mathbf{e}_j$$

$\Rightarrow$ $\mathcal{B}$ -matrix representation of T_A has the form

$$M = \begin{bmatrix} \lambda I_k & C \\ O_{(n-k)\times k} & D \end{bmatrix}$$

$\Rightarrow$ A and M are similar (Theorem 4.12 of Section 4.5).

$\Rightarrow$ A and M have the same characteristic polynomial, which is

$$
\begin{aligned}
\det\ (M - tI_n) &= \det \begin{bmatrix} \lambda I_k - tI_k & C \\ O & D - tI_{n-k} \end{bmatrix} \\
&= \det \begin{bmatrix} (\lambda - t)I_k & C \\ O & D - tI_{n-k} \end{bmatrix} \\
&= (\det(\lambda - t)I_k)\,(\det(D - tI_{n-k}) \\
&= (\lambda - t)^k\,(\det(D - tI_{n-k})
\end{aligned}
$$

$\Rightarrow$ algebraic multiplicity of $\lambda \geq k$.

Example:

$$
T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ 2x_1 - x_2 - x_3 \\ -x_3 \end{bmatrix} \xrightarrow[\text{matrix}]{\text{standard}} A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}
$$

$$
\xrightarrow[\text{polynomial}]{\text{characteristic}} \quad \det(A - tI_3) = (-1 - t)^3
$$

Problem Set for Sections 5.1 and 5.2

- Section 5.1: Problems 1, 3, 7, 9, 13, 15, 17, 19, 23, 25, 29, 31, 41, 43, 45, 47, 49, 51, 55, 57

- Section 5.2: Problems 1, 3, 5, 7, 9, 13, 15, 17, 21, 25, 27, 29, 33, 35, 39, 41, 53, 55, 57, 59, 61, 65, 69, 71.