CHAPTER 5

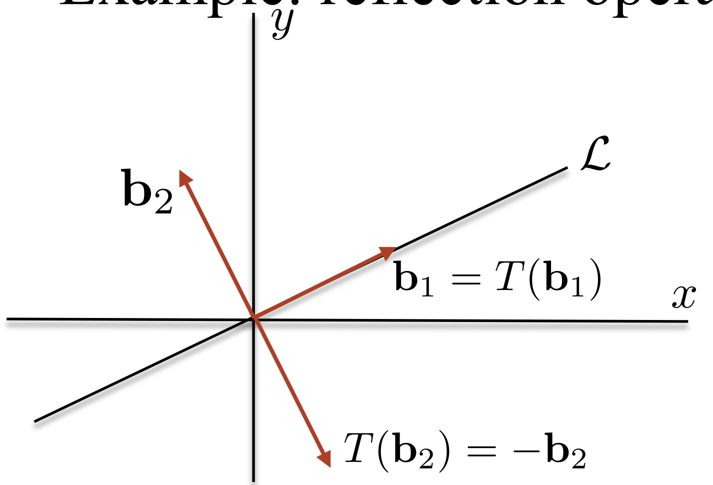
EIGENVALUES, EIGENVECTORS, AND DIAGONALIZATION

Definition

Let T be a linear operator on $\mathcal{R}^n$. A nonzero vector $\mathbf{v}$ in $\mathcal{R}^n$ is called an **eigenvector** of T if $T(\mathbf{v})$ is a multiple of $\mathbf{v}$, that is, $T(\mathbf{v}) = \lambda\mathbf{v}$ for some scalar λ. The scalar λ is called the **eigenvalue** of T that corresponds to $\mathbf{v}$.

Note: In these definitions $\mathbf{v} \in \mathcal{R}^n$ and $\lambda \in \mathcal{R}$, but sometimes it is necessary to extend the domain of T to allow $\mathbf{v} \in \mathcal{C}^n$ and $\lambda \in \mathcal{C}$.

Example: reflection operator T about the line $y = (1/2)x$



$\mathbf{b}_1$ is an eigenvector of T corresponding to the eigenvalue 1.

$\mathbf{b}_2$ is an eigenvector of T corresponding to the eigenvalue -1.

Definition

Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{v}$ in $\mathcal{R}^n$ is called an **eigenvector** of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ. The scalar λ is called the **eigenvalue** of A that corresponds to $\mathbf{v}$.

Note: In these definitions $\mathbf{v} \in \mathcal{R}^n$ and $\lambda \in \mathcal{R}$, but sometimes it is necessary to allow $\mathbf{v} \in \mathcal{C}^n$ and $\lambda \in \mathcal{C}$.

Example: $$A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$

$$A\begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix},\ A\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix},\ A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix},\ A\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

non-eigenvectors

eigenvectors

Example:

$$A = \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$A\mathbf{v} = \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 4\mathbf{v} \quad \Rightarrow$$

$$A(c\mathbf{v}) = c(A\mathbf{v}) = c(4\mathbf{v}) = 4(c\mathbf{v}), \forall c \in \mathcal{R}$$

An eigenvector of A corresponds to a unique eigenvalue.
An eigenvalue of A has infinitely many eigenvectors.

Property

The eigenvectors and corresponding eigenvalues of a linear operator are the same as those of its standard matrix.

Proof $T(\mathbf{v}) = \lambda\mathbf{v} \;\Leftrightarrow\; A\mathbf{v} = \lambda\mathbf{v}.$

Property

Let A be an $n \times n$ matrix with eigenvalue λ. The eigenvectors of A corresponding to λ are the nonzero solutions of $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$.

Proof $A\mathbf{v} = \lambda\mathbf{v} \Leftrightarrow A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \Leftrightarrow A\mathbf{v} - \lambda I_n\mathbf{v} = \mathbf{0} \Leftrightarrow (A - \lambda I_n)\mathbf{v} = \mathbf{0}$.

Definition

For $A \in \mathcal{R}^{n\times n}$, if λ is an eigenvalue of A, then the null space of $A - \lambda I_n$ is called the **eigenspace** of A corresponding to eigenvalue λ.

Definition

If λ is an eigenvalue of a linear operator T on $\mathcal{R}^n$, then the set of vectors $\mathbf{v}$ in $\mathcal{R}^n$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$ is called the **eigenspace** of T corresponding to eigenvalue λ.

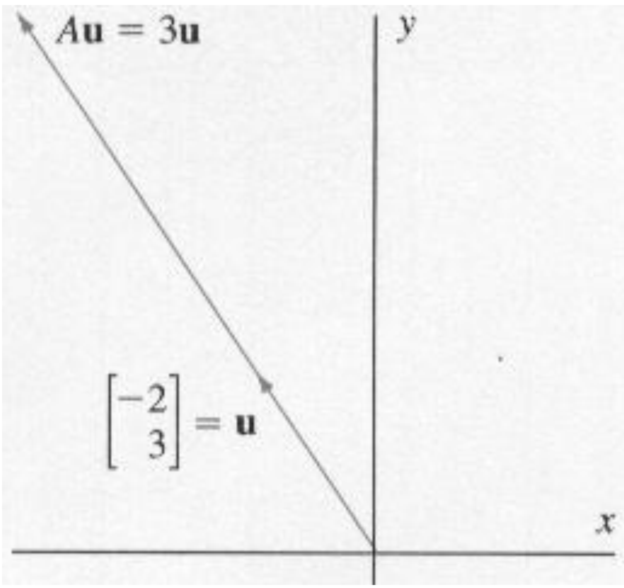
Example: to check 3 and -2 are eigenvalues of the linear operator

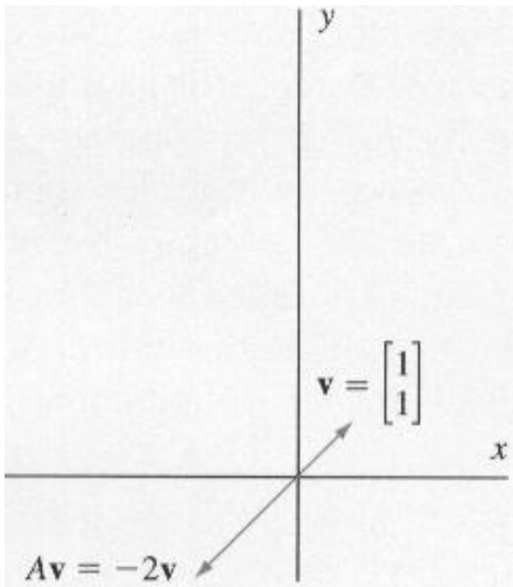
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_2 \\ -3x_1 + x_2 \end{bmatrix}$$

consider its standard matrix $A = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$

The rank of A - $3I_2$ is 1, so its null space is not the zero space, and every nonzero vector in the null space is an eigenvector of T corresponding to the eigenvalue 3.

Similarly, the rank of $A + 2I_2$ is 1, so its null space is not the zero space, and every nonzero vector in the null space is an eigenvector of T corresponding to the eigenvalue -2.





Example: to check that 3 is an eigenvalue of B and find a basis for the corresponding eigenspace, where

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

find the reduced row echelon form of B - $3I_3$ and the solution set of $(B - 3I_3)\mathbf{x} = \mathbf{0}$, respectively, to be

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Thus $\{[\ 1\ \ 0\ \ 0\ ]^T, [\ 0\ \ 1\ \ 1\ ]^T\}$ is a basis of the eigenspace of B corresponding to the eigenvalue 3.

Example: some square matrices and linear operators on $\mathcal{R}^n$ have no real eigenvalues, like the 90º-rotation matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$T(\mathbf{x}) = A\mathbf{x} = \lambda\mathbf{x}$$

An eig. vec. **x** ===> unique eig. val. λ.

An eig. val. λ ===> multiple eig. vectors **x**.

===> The set of all eig. vectors corr. to λ is

$$\underbrace{\{\mathbf{x} \in \mathcal{R}^n : (A - \lambda I_n)\mathbf{x} = \mathbf{0}\}}_{} \setminus \{\mathbf{0}\}$$

eigenspace corr. to the eig. val. λ

Homework Set for Sections 5.1

Section 5.1: Problems 1, 3, 7, 9, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59