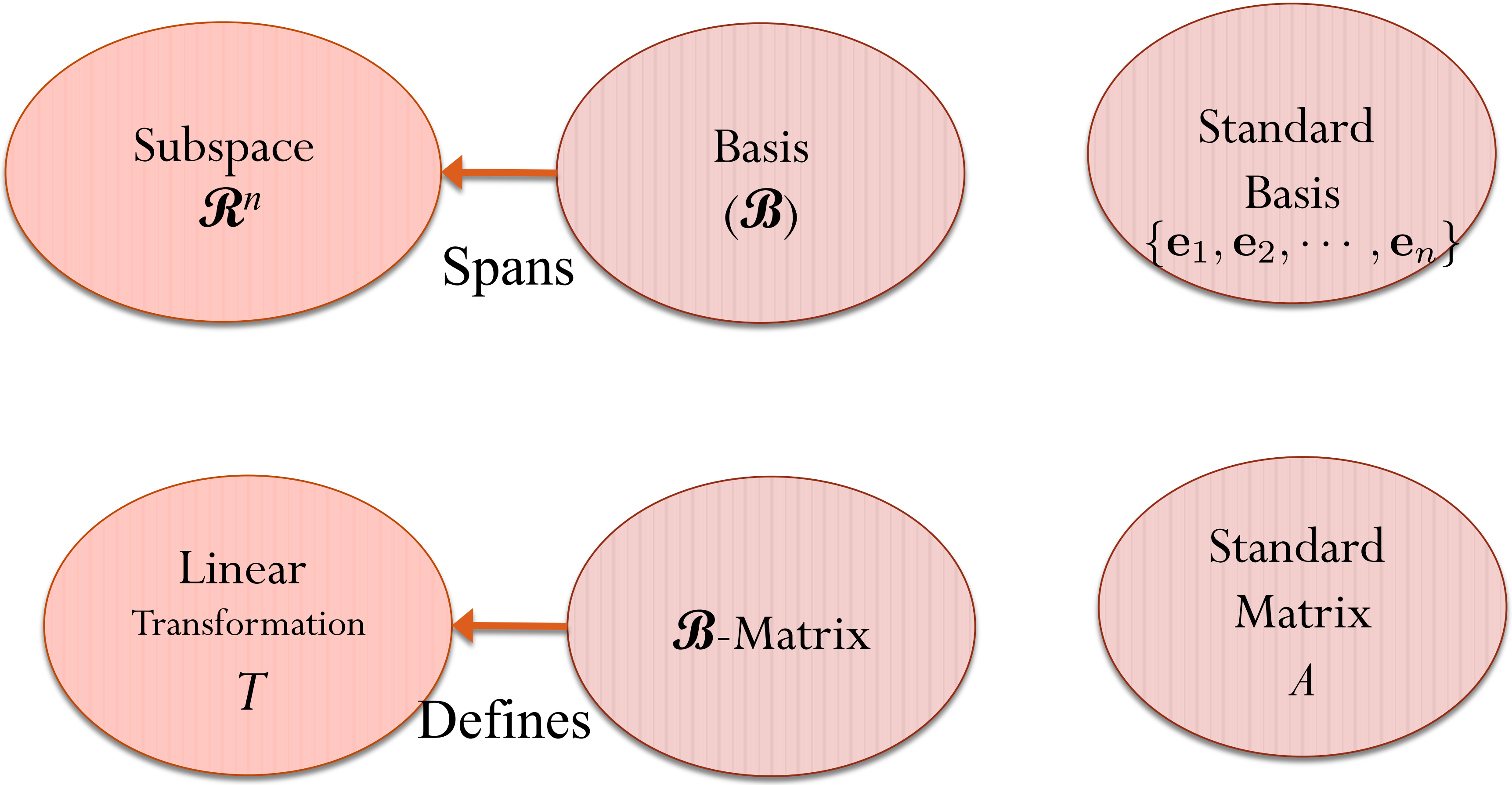
Vector spaces and Linear Transformations



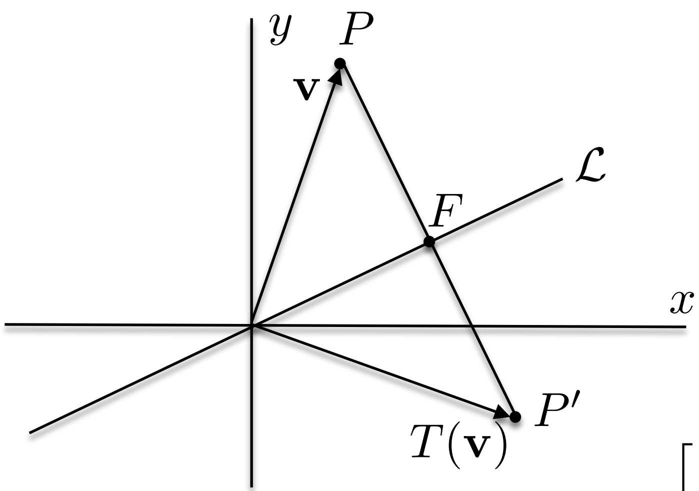
$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

Section 4.5 Matrix Representations of Linear Operators

Definition

A **linear operator** on $\mathcal{R}^n$ is a **linear transformation** from $\mathcal{R}^n$ to $\mathcal{R}^n$.

Example: reflection about a line $\mathcal{L}$ through the origin in $\mathcal{R}^2$



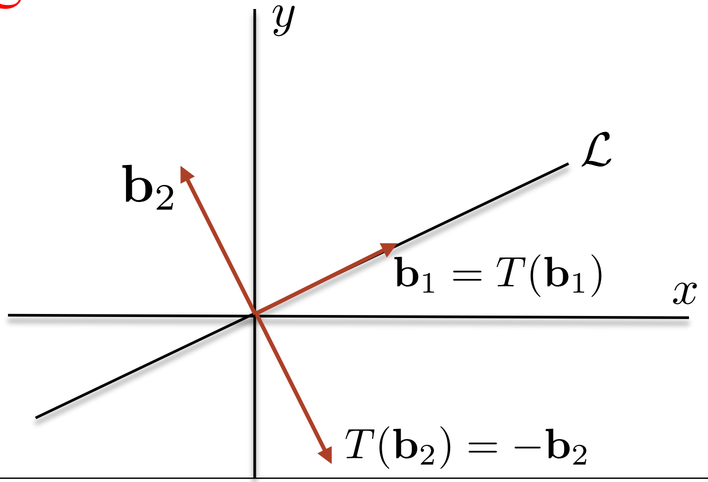
special case: $\mathcal{L}$ is the x-axis

$$T'\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

the standard matrix of T'

$$\begin{bmatrix} T'(\mathbf{e}_1) & T'(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & -\mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

general case:



consider the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

$$[T(\mathbf{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad [T(\mathbf{b}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Definition

Let T be a linear operator on $\mathcal{R}^n$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be a basis for $\mathcal{R}^n$. The matrix

$$\begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \end{bmatrix}$$

is called the **matrix representation** of T with respect to $\mathcal{B}$ or the **$\mathcal{B}$-matrix of T**. It is denoted $[T]_{\mathcal{B}}$.

Question

What is $[T]_{\mathcal{E}}$ where $\mathcal{E}$ is the standard basis for $\mathcal{R}^n$?

The Definition is an extension of the **standard matrix** to general basis. For $\mathcal{B} = \mathcal{E}$, the standard basis in $\mathcal{R}^n$,

$$[T]_{\mathcal{E}} = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{E}} & [T(\mathbf{b}_2)]_{\mathcal{E}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{E}} \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

Example: How do we find $[T]_{\mathcal{B}}$?

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_3 \\ x_1 + x_2 \\ -x_1 - x_2 + 3x_3 \end{bmatrix} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$$

How do we find $[T]_{\mathcal{B}}$?

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$$T(\mathbf{b}_1) = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, T(\mathbf{b}_2) = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}, T(\mathbf{b}_3) = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} \Rightarrow [T(\mathbf{b}_1)]_{\mathcal{B}} = B^{-1}T(\mathbf{b}_1) = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$[T(\mathbf{b}_2)]_{\mathcal{B}} = B^{-1}T(\mathbf{b}_2) = \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} \qquad [T(\mathbf{b}_3)]_{\mathcal{B}} = B^{-1}T(\mathbf{b}_3) = \begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix}$$

$$\Rightarrow [T]_{\mathcal{B}} = \begin{bmatrix} 3 & -9 & 8 \\ -1 & 3 & -3 \\ 1 & 6 & 1 \end{bmatrix}$$

Preview Question

Let T be a linear operator on $\mathcal{R}^n$ and A be the standard matrix of T.
Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be an (ordered) basis for $\mathcal{R}^n$ and
$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$.

Q: How can we write $[T]_{\mathcal{B}}$ in terms of matrices A and B?

Answer

1. No, it is not always possible to determine $[T]_{\mathcal{B}}$ from A and B?
2. $[T]_{\mathcal{B}} = B^{-1}A$?
3. $[T]_{\mathcal{B}} = AB^{-1}$?
4. $[T]_{\mathcal{B}} = B^{-1}AB$?
5. None of the above?

Theorem 4.12

Let T be a linear operator on $\mathcal{R}^n$, $\mathcal{B}$ a basis for $\mathcal{R}^n$, B the matrix whose columns are the vectors in $\mathcal{B}$, and A the standard matrix of T. Then $[T]_{\mathcal{B}} = B^{-1}AB$, or equivalently, $A = B[T]_{\mathcal{B}}B^{-1}$.

Proof Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$. Let $B = [\ \mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \cdots \ \ \mathbf{b}_n\ ]$.

$$\begin{aligned}
[T]_{\mathcal{B}} &= \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \end{bmatrix} \\
&= \begin{bmatrix} [A\mathbf{b}_1]_{\mathcal{B}} & [A\mathbf{b}_2]_{\mathcal{B}} & \cdots & [A\mathbf{b}_n]_{\mathcal{B}} \end{bmatrix} \\
&= \begin{bmatrix} B^{-1}(A\mathbf{b}_1) & B^{-1}(A\mathbf{b}_2) & \cdots & B^{-1}(A\mathbf{b}_n) \end{bmatrix} \\
&= \begin{bmatrix} (B^{-1}A)\mathbf{b}_1 & (B^{-1}A)\mathbf{b}_2 & \cdots & (B^{-1}A)\mathbf{b}_n \end{bmatrix} \\
&= B^{-1}A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} \\
&= B^{-1}AB.
\end{aligned}$$

$$\Rightarrow B[T]_{\mathcal{B}}B^{-1} = B(B^{-1}AB)B^{-1} = A.$$

Definition

For $A, B \in \mathcal{R}^{n\times n}$, if $B = P^{-1}AP$ for some invertible P, then A is **similar** to B.

Property: If A is similar to B, then B is similar to A. Thus we say A and B are similar.

Example:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_3 \\ x_1 + x_2 \\ -x_1 - x_2 + 3x_3 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$B = [\ \cdots\ ]$$

The standard matrix of T is

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix}$$

$$\Rightarrow [T]_{\mathcal{B}} = B^{-1}AB = \begin{bmatrix} 3 & -9 & 8 \\ -1 & 3 & -3 \\ 1 & 6 & 1 \end{bmatrix}$$

Example: Let T be a linear operator on $\mathscr{R}^3$

$$T\left(\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\mathbf{b}_1}\right)=\underbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}}_{\mathbf{c}_1}\qquad T\left(\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\mathbf{b}_2}\right)=\underbrace{\begin{bmatrix}3\\-1\\1\end{bmatrix}}_{\mathbf{c}_2}\qquad T\left(\underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{\mathbf{b}_3}\right)=\underbrace{\begin{bmatrix}2\\0\\1\end{bmatrix}}_{\mathbf{c}_3}$$

Q: Can T be uniquely determined?

$\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is L.I. $\Rightarrow$ $\mathscr{B}$ is a basis. Let $B = [\ \mathbf{b}_1\ \ \mathbf{b}_2\ \ \mathbf{b}_3\ ]$.

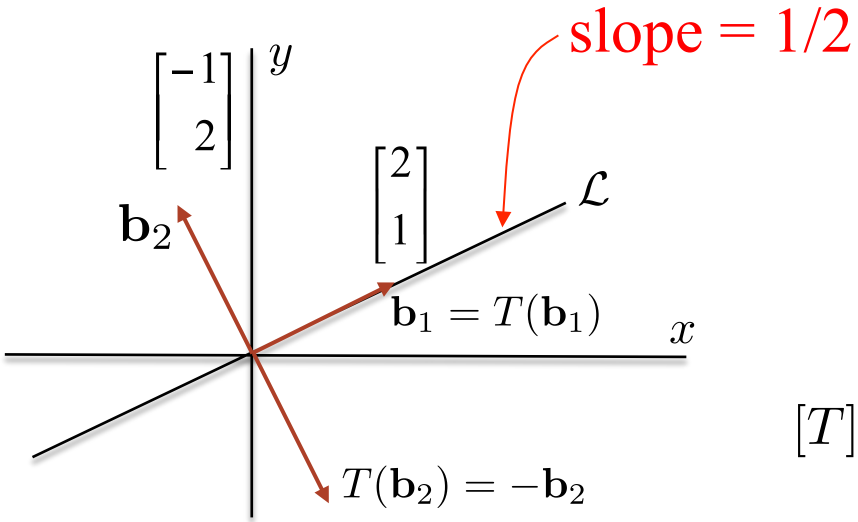
$$\begin{aligned}[T]_{\mathcal{B}} &= \left[\ [T(\mathbf{b}_1)]_{\mathcal{B}}\quad [T(\mathbf{b}_2)]_{\mathcal{B}}\quad [T(\mathbf{b}_3)]_{\mathcal{B}}\ \right]\\ &= \left[\ B^{-1}T(\mathbf{b}_1)\quad B^{-1}T(\mathbf{b}_2)\quad B^{-1}T(\mathbf{b}_3)\ \right]\\ &= \left[\ B^{-1}\mathbf{c}\quad B^{-1}\mathbf{c}_2\quad B^{-1}\mathbf{c}_3\ \right]\\ &= B^{-1}\left[\ \mathbf{c}_1\quad \mathbf{c}_2\quad \mathbf{c}_3\ \right].\end{aligned}$$

$$A = B[T]_{\mathcal{B}}B^{-1} = BB^{-1}\left[\ \mathbf{c}_1\quad \mathbf{c}_2\quad \mathbf{c}_3\ \right]B^{-1} = \left[\ \mathbf{c}_1\quad \mathbf{c}_2\quad \mathbf{c}_3\ \right]B^{-1}$$

$$= \begin{bmatrix}1 & 0 & 2\\ .5 & 1.5 & -1.5\\ -.5 & -.5 & 1.5\end{bmatrix}$$

Yes, T is uniquely determined by its images of a basis.

Example: reflection operator T about the line $y = (1/2)x$



Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $B = [\ \mathbf{b}_1 \ \ \mathbf{b}_2\ ]$, and A be the standard matrix of T.

$$[T]_{\mathcal{B}} = [\ [T(\mathbf{b}_1)]_{\mathcal{B}} \quad [T(\mathbf{b}_2)]_{\mathcal{B}}\ ] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow A = B[T]_{\mathcal{B}}B^{-1} = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .6x_1 + .8x_2 \\ .8x_1 - .6x_2 \end{bmatrix}$$

Homework Set for Sections 4.5

Section 4.5: Problems 1, 3, 7, 9, 13, 15, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37