# Section 4.4 Coordinate Systems

**Theorem 4.10**

Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ be a basis for a subspace $V$ of $\mathcal{R}^n$. Any vector $\mathbf{v}$ in $V$ can be uniquely represented as a linear combination of the vectors in $\mathcal{B}$; that is, there are unique scalars $a_1, a_2, \cdots, a_k$ such that $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k$.

***Proof*** Since $\mathcal{B}$ spans $V$, the existence of $a_1, a_2, \ldots, a_k$ is assured.
For any $b_1, b_2, \ldots, b_k$ making $\mathbf{v} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_k\mathbf{u}_k$, we have

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = (a_1 - b_1)\mathbf{u}_1 + (a_2 - b_2)\,\mathbf{u}_2 + \cdots + (a_k - b_k)\,\mathbf{u}_k$$

$\Rightarrow a_1 - b_1 = a_2 - b_2 = \cdots = a_k - b_k = 0$, since $\mathcal{B}$ is L.I..

## Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be a basis for $\mathcal{R}^n$. For each $\mathbf{v}$ in $\mathcal{R}^n$, there are unique scalars $c_1, c_2, \cdots, c_n$ such that $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$. The vector

ordered basis

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathcal{R}^n$$

is called the **coordinate vector** of $\mathbf{v}$ relative to $\mathcal{B}$ or the **$\mathcal{B}$-coordinate vector** of $\mathbf{v}$. We denote the $\mathcal{B}$-coordinate vector $\mathbf{v}$ by $[\mathbf{v}]_{\mathcal{B}}$.

Example:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$: a basis of $\mathcal{R}^3$.

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 6\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ 5 \end{bmatrix}$$

Example: For the same basis $\mathcal{B}$ above and $\mathbf{v} = [\ 1\ \text{-}4\ \ 4\ ]^T$, to find $[\mathbf{v}]_{\mathcal{B}}$, we need to determine the unique scalars $c_1$, $c_2$, $c_3$ such that

$$c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$$

$\Rightarrow c_1 = \text{-}6,\ c_2 = 4,\ c_3 = 3$ and $[\mathbf{v}]_{\mathcal{B}} = [\ \text{-}6\ \ 4\ \ 3\ ]^T$.

## Example

Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be the standard basis of $\mathcal{R}^n$. Then for all $\mathbf{v} \in \mathcal{R}^n$, $[\mathbf{v}]_{\mathcal{E}} =?$

## Example

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be an (ordered) basis of $\mathcal{R}^n$. Then for all $i = 1, 2, ..., n$, $[\mathbf{b}_i]_{\mathcal{B}} =?$

## Question

Let $\mathcal{B}$ be a basis for $\mathcal{R}^n$ and $B$ be the matrix whose columns are the vectors in $\mathcal{B}$.
How do we write $[\mathbf{v}]_{\mathcal{B}}$ in terms of $\mathbf{v}$ and $B$?

Example: Consider

$$\mathbf{v} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix} \text{ and } \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

## Theorem 4.11

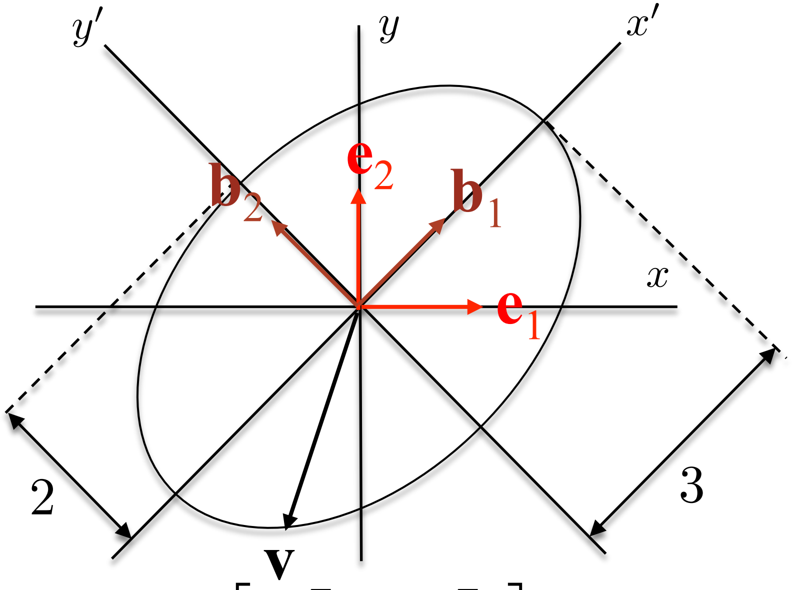
Let $\mathcal{B}$ be a basis for $\mathcal{R}^n$ and $B$ be the matrix whose columns are the vectors in $\mathcal{B}$. Then $B$ is invertible and, for every vector $\mathbf{v}$ in $\mathcal{R}^n$, $B[\mathbf{v}]_{\mathcal{B}} = \mathbf{v}$, or equivalently, $[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$.

***Proof*** Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, $\mathbf{v} \in \mathcal{R}^n$, and $[\mathbf{v}]_{\mathcal{B}} = [\ c_1\ \ c_2\ \cdots\ c_n\ ]^T$.

$$\Rightarrow \mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$
$$= [\ \mathbf{b}_1\ \ \mathbf{b}_2\ \ \cdots\ \ \mathbf{b}_n\ ]\,[\mathbf{v}]_{\mathcal{B}}$$
$= B[\mathbf{v}]_{\mathcal{B}}$, where $B$ is invertible, since it has $n$ L.I. columns.

Example: $\mathbf{v} = \begin{bmatrix} 1 & -4 & 4 \end{bmatrix}^T$ and

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\} \Rightarrow [\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}$$




$$A_{45^\circ} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

The equation of the ellipse in the $x', y'$-coordinate system is

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

Consider the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

$$\mathbf{b}_1 = A_{45^\circ}\mathbf{e}_1 \qquad \mathbf{b}_2 = A_{45^\circ}\mathbf{e}_2$$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} A_{45^\circ}\mathbf{e}_1 & A_{45^\circ}\mathbf{e}_2 \end{bmatrix} = A_{45^\circ}\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = A_{45^\circ} I_2 = A_{45^\circ}$$

$$[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v} = B^T\mathbf{v}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = B^T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \\ -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{bmatrix}$$

Thus, the equation of the ellipse in the *x*, *y*-coordinate system is

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

$$\Rightarrow \frac{\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{3^2} + \frac{\left(-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{2^2} = 1$$

or $\quad 13x^2 - 10xy + 13y^2 = 72$

## Example

Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be the standard basis of $\mathcal{R}^n$. Then for all $\mathbf{v} \in \mathcal{R}^n$, $[\mathbf{v}]_{\mathcal{E}} =$?

(1) $\mathbf{e}_1$; (2) $\mathbf{e}_2$; (3) $\mathbf{e}_n$; (4) $\mathbf{v}$; (5) None of the above

## Example

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be an (ordered) basis of $\mathcal{R}^n$. Then for all $i = 1, 2, ..., n$, $[\mathbf{b}_i]_{\mathcal{B}} =$?

(1) $\mathbf{b}_1$; (2) $\mathbf{b}_i$; (3) $\mathbf{e}_i$; (4) $\mathbf{e}_n$; (5) None of the above

## Homework Set for Section 4.4

Section 4.4: Problems 1, 5, 7, 9, 11, 13, 15, 19, 23, 27, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49