Section 4.2 Basis and Dimension

In Sec. 4.1, we learned that the span of any nonempty subset of $\mathcal{R}^n$ is a subspace of $\mathcal{R}^n$.

Question 1: Reversely, can any subspace of $\mathcal{R}^n$ be written as the span of a subset of $\mathcal{R}^n$?

Question 2: If so, is there a "best criterion" to choose such a subset that "spans" (generates) such a subspace of $\mathcal{R}^n$?

Section 4.2 Basis and Dimension

Definition

Let V be a nonzero subspace of $\mathcal{R}^n$. A **basis** for V is a **linearly independent generation set** for V.

Example: the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathcal{R}^n$.

The pivot columns of a matrix form a basis for its column space.

Proof By Theorem 2.4 of Section 2.3.

Property

The pivot columns of a matrix form a basis for its column space.

Example:

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \xrightarrow[\text{echelon form}]{\text{reduced row}} \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

$$\Rightarrow \text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Theorem 4.3 (Reduction Theorem)

Let S be a finite generating set for a nonzero subspace V of $\mathcal{R}^n$. Then S can be reduced to a basis for V by removing vectors from S.

Proof Suppose $\boldsymbol{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$. Let $A = [\, \mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \cdots \;\; \mathbf{u}_k \,]$.

$\Rightarrow$ Col A = Span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} = V$, and the set of pivot columns of A, which is a subset of $\boldsymbol{S}$, form a basis of V.

Theorem 4.3 (Reduction Theorem)

Let S be a finite generating set for a nonzero subspace V of $\mathcal{R}^n$. Then S can be reduced to a basis for V by removing vectors from S.

Corollary 1: If S is a finite generating set of $\mathcal{R}^n$, then S has a subset forming a basis of $\mathcal{R}^n$.

Corollary 2: Any generating set, and hence every basis, of $\mathcal{R}^n$ must contain at least n vectors.

Proof Following the proof of the main theorem, Col $A = V = \mathcal{R}^n$.

$\Rightarrow$ rank $A = n$ by Theorem 1.6 $\Rightarrow k \geq n$.

Every subset of $\mathcal{R}^n$ with more than n vectors is L.D. (Section 1.7).

$\Rightarrow$ Every basis of $\mathcal{R}^n$ contains at most n vectors.

$\Rightarrow$ Every basis of $\mathcal{R}^n$ contains exactly n vectors.

Theorem 4.4 (Extension Theorem)

Let S be a linearly independent subset of a nonzero subspace V of $\mathcal{R}^n$. Then S can be extended to a basis for V by inclusion of additional vectors. In particular, every nonzero subspace has a basis.

Proof If Span$\boldsymbol{S} = V$, then $\boldsymbol{S}$ is a basis itself.

If Span$\boldsymbol{S} \neq V$, then $\exists\ \mathbf{v}_1 \in V \setminus \text{Span}\ \boldsymbol{S}$, and $\boldsymbol{S} \cup \{\mathbf{v}_1\}$ is L.I..

If Span$[\boldsymbol{S} \cup \{\mathbf{v}_1\}] = V$, then $\boldsymbol{S} \cup \{\mathbf{v}_1\}$ is a basis and $\boldsymbol{S} \subseteq \boldsymbol{S} \cup \{\mathbf{v}_1\}$.

If Span$[\boldsymbol{S} \cup \{\mathbf{v}_1\}] \neq V$, then $\exists\ \mathbf{v}_2 \in V \setminus \text{Span}[(\boldsymbol{S} \cup \{\mathbf{v}_1\})]$, and $\boldsymbol{S} \cup \{\mathbf{v}_1, \mathbf{v}_2\}$ is L.I..

If $\cdots \Rightarrow \exists\ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ such that $\boldsymbol{S} \cup \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is L.I. and Span$[\boldsymbol{S} \cup \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}] = V$, where $k \leq n$.

$\Rightarrow \boldsymbol{S} \cup \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis and $\boldsymbol{S} \subseteq \boldsymbol{S} \cup \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

In particular, let V be a nonzero subspace. $\Rightarrow \exists\ \mathbf{u} \neq \mathbf{0}$ in V.

$\Rightarrow$ can start from the L.I. subset $\boldsymbol{S} = \{\mathbf{u}\}$ to get a basis for V.

Theorem 4.5

Let V be a nonzero subspace of $\mathcal{R}^n$. Then any two bases of V contain the same number of vectors.

Proof Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ are two bases of V.

Let $A = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_k]$ and $B = [\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_p]$.

Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ spans V, $\exists\ \mathbf{c}_i \in \mathcal{R}^k$ s.t. $A\mathbf{c}_i = \mathbf{v}_i$ for all i

Reversing the roles of the two bases one has $k \le p \Rightarrow p = k$.

Theorem 4.5

Let V be a nonzero subspace of $\mathcal{R}^n$. Then any two bases of V contain the same number of vectors.

Definition

The number of vectors in a basis for a subspace V of $\mathcal{R}^n$ is called the **dimension** of V and is denoted $\dim V$. Also, $\dim \{\mathbf{0}\} = 0$.

A basis is
1) a generating set for a subspace with the fewest possible vectors;
2) a L.I. subset of a subspace that is as large as possible.

Definition

The number of vectors in a basis for a subspace V of $\mathcal{R}^n$ is called the **dimension** of V and is denoted $\dim V$. Also, $\dim \{\mathbf{0}\} = 0$.

Example:

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathcal{R}^4 : x_1 - 3x_2 + 5x_3 - 6x_4 = 0 \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 + 6x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$S = \{\cdots\}$: L.I. and generating V. $\Rightarrow$ dim. $V = 3$.

Example: Is $\mathcal{C}$ a basis of V?

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathcal{R}^3 : v_1 - v_2 + 2v_3 = 0 \right\}$$

1. Check that both vectors in $\mathcal{C}$ belong to V:

$$1 - 1 + 2(0) = 0 \text{ and } -1 - 1 + 2(1) = 0$$

2. Check that $\mathcal{C}$ is L.I.: neither is a multiple of each other.
3. Check that $\mathcal{C}$ spans V: $\exists$ c_1 and c_2 such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Or use the following Theorem (will be proved in the next slide):

Theorem 4.6

Let V be a subspace of $\mathcal{R}^n$ with dimension k. Then every **linearly independent** subset of V contains at most k vectors; or equivalently, any finite subset of V containing more than k vectors is **linearly dependent**.

Theorem 4.6

Let V be a subspace of $\mathcal{R}^n$ with dimension k. Then every **linearly independent** subset of V contains at most k vectors; or equivalently, any finite subset of V containing more than k vectors is **linearly dependent**.

Proof Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a L.I. subset of V. By the Extension Theorem, this set can be extended to a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \ldots, \mathbf{v}_k\}$ for V. It follows that $p \leq k$.

Theorem 4.7

Let V be a k-dimensional subspace of $\mathcal{R}^n$. Suppose that S is a subset of V with exact k vectors. Then S is a basis for V if either S is linearly independent or S is a generating set for V.

Proof Let $\mathcal{S} \subseteq V$ be L.I. and have k vectors.

$\Rightarrow$ By the Extension Theorem, $\exists$ a basis $\mathcal{B}$ s.t. $\mathcal{S} \subseteq \mathcal{B}$.

$\Rightarrow$ $\mathcal{B}$ has k vectors, since dim. $V = k$.

$\Rightarrow$ $\mathcal{B} = \mathcal{S}$.

Let $\mathcal{S} \subseteq V$ be a generating set for V and have k vectors.

$\Rightarrow$ By the Reduction Theorem, $\exists$ a basis $\mathcal{B} \subseteq \mathcal{S}$.

$\Rightarrow$ $\mathcal{B}$ has k vectors, since dim. $V = k$.

$\Rightarrow$ $\mathcal{B} = \mathcal{S}$.

Theorem 4.7

Let V be a k-dimensional subspace of $\mathcal{R}^n$. Suppose that S is a subset of V with exact k vectors. Then S is a basis for V if either S is linearly independent or S is a generating set for V.

Example: Is $\boldsymbol{\mathcal{B}}$ a basis of V?

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\} V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \mathcal{R}^4 : v_1 + v_2 + v_4 = 0 \right\}$$

$$\left.\begin{array}{l} \text{1. Check that all vectors in } \boldsymbol{\mathcal{B}} \text{ belong to } V. \\ \text{2. } \boldsymbol{\mathcal{B}} \text{ is L.I. (you check it).} \\ \text{3. dim. } V = 3 \text{ (4 variables, 1 nonzero equation.)} \end{array}\right\} \boldsymbol{\mathcal{B}} \text{ is a basis of } V.$$

Example: Is $\mathcal{B}$ a basis of W = Span $\mathcal{S}$?

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Let $B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 2 & -1 & 1 & -1 \end{bmatrix}$

1. $\mathcal{B} \subseteq W$ (you check that rank A = rank $[\ A \ \ B\ ]$.)
2. $\mathcal{B}$ is L.I. (you check it).
3. dim. $W = 3$ (you check that rank $A = 3$.)

$\Rightarrow$ $\mathcal{B}$ is a basis of W.

Homework Set for Sections 4.1 and 4.2

Section 4.2: Problems 7, 13, 22, 36, 39-44