# Main concepts in Chapter 4

Subspace

Basis

Dimension

# Main concepts in Chapter 1

Linear Independence

Span

Rank

Linear Combination

(Columns of) Matrices

**Set**
(of vectors)

(Vector)
**Space**

Sub**set**

Sub**space**

# CHAPTER 4
# SUBSPACES AND THEIR PROPERTIES

## Definition. (Subspace)

A subset $W$ of $\mathcal{R}^n$ is called a **subspace** of $\mathcal{R}^n$ if it has the following three properties.

1. The zero vector belongs to $W$.
2. Whenever $\mathbf{u}$ and $\mathbf{v}$ belong to $W$, then $\mathbf{u}+\mathbf{v}$ belongs to $W$. (In this case, we say that $W$ is **closed under (vector) addition**).
3. Whenever $\mathbf{u}$ belongs to $W$ and $c$ is a scalar, then $c\mathbf{u}$ belongs to $W$. (In this case, we say that $W$ is **closed under scalar multiplication**)

Example: non-subspaces

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{R}^2 : w_1 \geq 0 \text{ and } w_2 \geq 0 \right\} \qquad \mathbf{u} \in \mathcal{S}_1,\ \mathbf{u} \neq \mathbf{0} \Rightarrow -\mathbf{u} \notin \mathcal{S}_1$$

$$\mathcal{S}_2 = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{R}^2 : w_1^2 = w_2^2 \right\} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in \mathcal{S}_2 \text{ but } \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin \mathcal{S}_2$$

Example: $\mathcal{R}^n$ and $\{\mathbf{0}\}$ are both subspaces of $\mathcal{R}^n$.

$\{\mathbf{0}\}$ is the zero subspace, and others are non-zero subspaces.

Example:

$$W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathcal{R}^3 : 6w_1 - 5w_2 + 4w_3 = 0 \right\}$$ is a subspace, since

1. $\mathbf{0} \in W$ (for $\mathbf{0} = [\ 0\ \ 0\ \ 0\ ]^T$, $6(0) - 5(0) + 4(0) = 0$);
2. $\mathbf{u} = [\ u_1\ \ u_2\ \ u_3\ ]^T$, $\mathbf{v} = [\ v_1\ \ v_2\ \ v_3\ ]^T \in W \Rightarrow \mathbf{u}+\mathbf{v} \in W$
   $(6(u_1+v_1) - 5(u_2+v_2) + 4(u_3+v_3)$
   $= 6u_1 - 5u_2 + 4u_3 + 6v_1 - 5v_2 + 4v_3 = 0 + 0 = 0)$;
3. $\mathbf{u} = [\ u_1\ \ u_2\ \ u_3\ ]^T \in W \Rightarrow c\mathbf{u} \in W$
   $(6(cu_1) - 5(cu_2) + 4(cu_3) = c(6u_1 - 5u_2 + 4u_3) = c0 = 0)$.

Example: $W = \{c\mathbf{w} \mid c \in \mathcal{R}\}$ is a subspace (you show it).

## Theorem 4.1

The span of a finite nonempty subset of $\mathcal{R}^n$ is a subspace of $\mathcal{R}^n$.

***Proof*** Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$ and $W = \text{Span } S$.

1. $\mathbf{0} \in W$.

2. $\mathbf{u} + \mathbf{v} \in W$ for any $\mathbf{u}, \mathbf{v} \in W$.

3. $c\mathbf{u} \in W$ for any $\mathbf{u} \in W$ and $c \in \mathcal{R}$.

**Theorem 4.1**

The span of a finite nonempty subset of $\mathcal{R}^n$ is a subspace of $\mathcal{R}^n$.

Example:

$$W = \left\{ \begin{bmatrix} 2a - 3b \\ b \\ -a + 4b \end{bmatrix} \in \mathcal{R}^3 : a \in \mathcal{R} \text{ and } b \in \mathcal{R} \right\} = \text{Span} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$$

$\Rightarrow$ $W$ is a subspace of $\mathbb{R}^3$.

## Definition

The **null space** of a matrix $A$ is the solution set of $A\mathbf{x} = \mathbf{0}$. It is denoted Null $A$.

$A \in \mathcal{R}^{m\times n} \Rightarrow \text{Null } A = \{\, \mathbf{v} \in \mathcal{R}^n : A\mathbf{v} = \mathbf{0} \,\}$, the solution set of the homogeneous system of linear equations $A\mathbf{v} = \mathbf{0}$.

## Theorem 4.2

If $A$ is an $m \times n$ matrix, then Null $A$ is a subspace of $\mathcal{R}^n$.

***Proof***

Example:

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 3 \\ -1 \end{bmatrix} \in \text{Null } A? \quad \mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 2 \\ 1 \end{bmatrix} \in \text{Null } A?$$

parametric form of the general solution to $A\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 4x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

$$\text{Null } A = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

$A\mathbf{u} = \mathbf{0}$

$\Rightarrow \mathbf{u} \in \text{Null } A$

$A\mathbf{v} = [\ 0\ \ \text{-}10\ \ 10\ ]^T \neq \mathbf{0}$

$\Rightarrow \mathbf{v} \notin \text{Null } A$

## Definition

The **column space** of a matrix $A$ is the span of its columns. It is denoted Col $A$.

$$A \in \mathcal{R}^{m\times n} \Rightarrow \text{Col } A = \{A\mathbf{v} : \mathbf{v} \in \mathcal{R}^n\}$$

## Definition

For $A \in \mathcal{R}^{m\times n}$, the **row space** of $A$, denoted as Row $A$, is the subspace of $\mathcal{R}^m$ spanned by the rows of $A$.

$$\text{Row } A = \text{Col } A^T$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \in \text{Col } A? \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \in \text{Col } A?$$

The reduced row echelon form of $[\, A \;\; \mathbf{u} \,]$ is $\begin{bmatrix} 1 & 2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow A\mathbf{x} = \mathbf{u}$ is inconsistent, and $\mathbf{u} \notin \text{Col } A$.

The reduced row echelon form of $[\, A \;\; \mathbf{v} \,]$ is $\begin{bmatrix} 1 & 2 & 0 & -4 & 0.5 \\ 0 & 0 & 1 & 3 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow A\mathbf{x} = \mathbf{v}$ is consistent, and $\mathbf{v} \in \text{Col } A$.

The range of a linear transformation is the same as the column space of its standard matrix.

Example:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 - x_4 \\ 2x_1 + 4x_2 - 8x_4 \\ 2x_3 + 6x_4 \end{bmatrix}$$ has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix} \Rightarrow \text{Range of } T = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -8 \\ 6 \end{bmatrix}\right\}$$

The null space of a linear transformation is the null space of its standard matrix.

Example:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 - x_4 \\ 2x_1 + 4x_2 - 8x_4 \\ 2x_3 + 6x_4 \end{bmatrix}$$ has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$, for which the reduced row echelon form is

$$\begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Null space of } T = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}\right\}$$

## Homework Set for Section 4.1

Section 4.1: Problems 2, 8, 11, 19, 27, 29, 35, 37, 77, 78, 80