

Definition. (Submatrix for cofactor)

Suppose $A = [a_{ij}] \in \mathcal{M}_{n \times n}$ is an $n \times n$ square matrix.

An $(n - 1) \times (n - 1)$ matrix A_{ij} is defined as the submatrix A obtained by removing the i th row and the j th column of A .

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

i th row
 j th column

Definition. (Determinants and Cofactors)

Suppose $A = [a_{ij}] \in \mathcal{M}_{n \times n}$ is an $n \times n$ square matrix.

The **determinant** of A , denoted by $\det A$ or $|A|$, is defined as $\det A = a_{11}$ for $n = 1$ and

$$\det A = a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

1 for $n > 1$. The (i, j) -**cofactor** c_{ij} of A is defined as $\underline{(-1)^{i+j}} \underline{\det A_{ij}}$.

Question

Consider two $n \times n$ matrices A and B . Is

$$\det (AB) = \det A \cdot \det B$$

always true?

Theorem 3.3 (a)

Let A be an $n \times n$ matrix. If B is a matrix obtained by interchanging two rows of A , then $\det B = -\det A$.

Proof If B is obtained by interchanging row r and row $s = r + 1$ of A ,



$$\Rightarrow a_{rj} = b_{sj} \text{ and } A_{rj} = B_{sj} \quad \forall j.$$

$$\begin{aligned} \Rightarrow & (-1)^{r+1} a_{r1} \det A_{r1} + (-1)^{r+2} a_{r2} \det A_{r2} + \cdots + (-1)^{r+n} a_{rn} \det A_{rn} \\ &= (-1)^{r+1} b_{s1} \det B_{s1} + (-1)^{r+2} b_{s2} \det B_{s2} + \cdots + (-1)^{r+n} b_{sn} \det B_{sn} \\ &= -(-1)^{s+1} b_{s1} \det B_{s1} - (-1)^{s+2} b_{s2} \det B_{s2} - \cdots - (-1)^{s+n} b_{sn} \det B_{sn} \end{aligned}$$

$$\Rightarrow \det A = -\det B.$$

If B is obtained by interchanging row r and row $s > r + 1$ of A ,

$\Rightarrow B$ may be obtained from A by making adjacent row interchanges:

 $S - r$ adjacent row interchanges	\vdots $\cdots \mathbf{a}'_{S-1} \cdots$ $\cdots \mathbf{a}'_S \cdots$ $\cdots \mathbf{a}'_r \cdots$ \vdots	 $S - r - 1$ adjacent row interchanges
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Theorem 3.3 (b)

Let A be an $n \times n$ matrix. If B is a matrix obtained by multiplying each entry of some row of A by a scalar k , then $\det B = k \det A$.

Proof (b) If B is obtained by multiplying row r of A by k ,

$$\Rightarrow \det B$$

$$= (-1)^{r+1} b_{r1} \det B_{r1} + (-1)^{r+2} b_{r2} \det B_{r2} + \cdots + (-1)^{r+n} b_{rn} \det B_{rn}$$

$$= (-1)^{r+1} k a_{r1} \det A_{r1} + (-1)^{r+2} k a_{r2} \det A_{r2} + \cdots + (-1)^{r+n} k a_{rn} \det A_{rn}$$

$$= k \cdot \det A.$$

Theorem 3.3 (c)

Let A be an $n \times n$ matrix. If B is a matrix obtained by adding a multiple of some row of A to a different row, then $\det B = \det A$.

Proof If $C \in \mathcal{M}_{n \times n}$ has two identical rows, then by (b) $\det C = -\det C$, since $C = C$ with the two identical rows interchanged.
 $\Rightarrow \det C = 0$.

If B is obtained by adding k times row s of A to row r ($\neq s$),

$$\begin{aligned}\Rightarrow \det B &= (-1)^{r+1} b_{r1} \det B_{r1} + \cdots + (-1)^{r+n} b_{rn} \det B_{rn} \\&= (-1)^{r+1} (a_{r1} + ka_{s1}) \det A_{r1} + \cdots + (-1)^{r+n} (a_{rn} + ka_{sn}) \det A_{rn} \\&= (-1)^{r+1} a_{r1} \det A_{r1} + \cdots + (-1)^{r+n} a_{rn} \det A_{rn} \\&\quad + k \cdot [(-1)^{r+1} a_{s1} \det A_{r1} + \cdots + (-1)^{r+n} a_{sn} \det A_{rn}] \\&= \det A + k \cdot \det C, \text{ where rows } r \text{ and } s \text{ of } C \text{ are identical.} \\&\Rightarrow \det B = \det A + k \cdot 0 = \det A\end{aligned}$$

Theorem 3.3

Let A be an $n \times n$ matrix.

- (a) If B is a matrix obtained by interchanging two rows of A , then $\det B = -\det A$.
- (b) If B is a matrix obtained by multiplying each entry of some row of A by a scalar k , then $\det B = k \det A$.
- (c) If B is a matrix obtained by adding a multiple of some row of A to a different row, then $\det B = \det A$.
- (d) For any $n \times n$ elementary matrix E , we have $\det EA = (\det E)(\det A)$.

Proof (d) If $E \in \mathcal{M}_{n \times n}$ is an elementary matrix obtained by interchanging two rows of I_n , then $\det E = -\det I_n = -1$, and by (a) $\det EA = -\det A = (\det E)(\det A)$. For the other two types of elementary matrices, the proofs are similar.

With the steps 1-4 of the Gaussian elimination algorithm, every $A \in \mathcal{R}^{n \times n}$ may be transformed into a row echelon form by using elementary row operations **other than scaling operations**.

Matrices in the row echelon form are **upper triangular**.

If an $n \times n$ matrix A is transformed into an upper triangular matrix U by elementary row operations other than scaling operations, then

$$\det A = (-1)^r u_{11} u_{22} \cdots u_{nn},$$

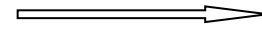
where r is the number of row interchanges performed.

Proof $E_k \cdots E_2 E_1 A = U \Rightarrow \det(E_k) \cdots \det(E_2) \det(E_1) \det(A) = \det U$.

Since $\det(E_k) = \pm 1$, we have $(-1)^r \det A = \det U$.

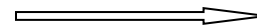
Example:

$$A = \begin{bmatrix} 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ -2 & 0 & 4 & -7 \\ 4 & -4 & 4 & 15 \end{bmatrix}$$



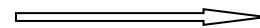
$$\begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 0 & 4 & -2 \\ 0 & 1 & 3 & -3 \\ 4 & -4 & 4 & 15 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 0 & 4 & -2 \\ 0 & 1 & 3 & -3 \\ 0 & -4 & 12 & 1 \end{bmatrix}$$



$$\begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & -4 & 12 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 24 & -11 \end{bmatrix}$$



$$\begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

$$\Rightarrow \det A = (-1)^2 \det U = (-1)^2 \cdot (-2) \cdot 1 \cdot 4 \cdot 1 = -8.$$

Using Gaussian elimination to evaluate determinants is much faster than using cofactor expansion, especially for large matrices.

For any $A \in \mathcal{M}_{n \times n}$, A is not invertible if and only if $\det A = 0$.

Proof $\text{rank } A < n \Leftrightarrow$ its row echelon form has the zero bottom row.

Theorem 3.4 (a)(b)

Let A and B be square matrices of the same size. The following statements are true.

- (a) A is invertible if and only if $\det A \neq 0$.
- (b) $\det AB = (\det A)(\det B)$.

Proof

b) If A is invertible, then \exists elementary matrices E_1, E_2, \dots, E_k , such that $A = E_k \cdots E_2 E_1$.

$$\begin{aligned} \Rightarrow (\det A)(\det B) &= (\det E_k) \cdots (\det E_2) (\det E_1)(\det B) \\ &= (\det E_k) \cdots (\det E_2)(\det E_1 B) = \cdots \\ &= \det(E_k \cdots E_2 E_1 B) = \det AB. \end{aligned}$$

If A is not invertible, then \exists an invertible P such that $PA = R$, the reduced row echelon form of A .

$\Rightarrow R$, and thus RB , have the zero bottom rows.

$$\Rightarrow (\det P)(\det AB) = \det P(AB) = \det RB = 0 \Rightarrow \det AB = 0,$$

but $(\det A)(\det B) = 0 \cdot (\det B) = 0$.

Theorem 3.4 (c)

Let A be a square matrix. Then $\det A^T = \det A$.

Proof

(c) If A is invertible, then \exists elementary matrices E_1, E_2, \dots, E_k , such that $A = E_k \cdots E_2 E_1$, and $A^T = E_1^T E_2^T \cdots E_k^T$.

$$\begin{aligned}\Rightarrow \det A^T &= \det(E_1^T E_2^T \cdots E_k^T) = (\det E_1^T)(\det E_2^T) \cdots (\det E_k^T) \\ &= (\det E_1)(\det E_2) \cdots (\det E_k) \\ &= (\det E_k) \cdots (\det E_2)(\det E_1) \\ &= \det(E_k \cdots E_2 E_1) = \det A.\end{aligned}$$

If A is not invertible, then A^T is not invertible, otherwise $(A^T)^T = A$ would be invertible.

$$\Rightarrow \det A^T = 0 = \det A.$$

Theorem 3.4 (d)

If A is an invertible matrix. Then $\det A^{-1} = \frac{1}{\det A}$.

Proof

Theorem 3.4

Let A and B be square matrices of the same size. The following statements are true.

- (a) A is invertible if and only if $\det A \neq 0$.
- (b) $\det AB = (\det A)(\det B)$.
- (c) $\det A^T = \det A$.
- (d) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.

Example:

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & c \\ 2 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & c+2 \\ 2 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & c+2 \\ 0 & 3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & c+2 \\ 0 & 0 & 3c+9 \end{bmatrix}$$

\therefore The matrix is invertible if and only if $c \neq -3$.

Example:

$$M = \begin{bmatrix} A & B \\ O & C \end{bmatrix} \begin{matrix} \underbrace{\hspace{1cm}}_m \\ \underbrace{\hspace{1cm}}_n \end{matrix} \begin{matrix} \underbrace{\hspace{1cm}}_m \\ \underbrace{\hspace{1cm}}_n \end{matrix} = \begin{bmatrix} I_m & O' \\ O & C \end{bmatrix} \begin{bmatrix} A & B \\ O & I_n \end{bmatrix}$$

m × n

$$\Rightarrow \det M = \det \begin{bmatrix} I_m & O' \\ O & C \end{bmatrix} \cdot \det \begin{bmatrix} A & B \\ O & I_n \end{bmatrix} = (\det C)(\det A).$$

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \det(A + B) \neq \det A + \det B$$

Homework Set for Section 3.2

Section 3.2: Problems 7, 13, 22, 36, 39-44