Definition. (Submatrix for cofactor)

Suppose $A = [a_{ij}] \in \mathcal{M}_{n\times n}$ is an $n \times n$ square matrix.
An $(n-1)\times(n-1)$ matrix A_{ij} is defined as the submatrix A obtained by removing the ith row and the jth column of A.

$$A_{ij} = \begin{array}{cc} \left[\begin{array}{ccccc} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{array}\right] & \begin{array}{l} \\ \\ i\text{th row} \\ \\ \\ \end{array} \\ \begin{array}{ccccc} & & j\text{th column} & & \end{array} & \end{array}$$

Definition. (Determinants and Cofactors)

Suppose $A = [a_{ij}] \in \mathcal{M}_{n\times n}$ is an $n \times n$ square matrix.
The **determinant** of A, denoted by $\det\ A$ or $|A|$, is defined as $\det\ A = a_{11}$ for $n = 1$ and

$$\det\ A = a_{11}\cdot\det\ A_{11} - a_{12}\cdot\det\ A_{12} + \cdots + (-1)^{1+n}a_{1n}\det\ A_{1n}$$

for $n > 1$. The **(i,j)-cofactor** c_{ij} of A is defined as $(-1)^{i+j}\det\ A_{ij}$.

Question

Consider two $n \times n$ matrices A and B. Is

$$\det(AB) = \det A \cdot \det B$$

always true?

Theorem 3.3 (a)

Let A be an $n \times n$ matrix. If B is a matrix obtained by interchanging two rows of A, then $\det B = -\det A$.

Proof If B is obtained by interchanging row r and row $s = r + 1$ of A,

$$\Rightarrow a_{rj} = b_{sj} \text{ and } A_{rj} = B_{sj} \ \forall j.$$

$$\Rightarrow (-1)^{r+1} a_{r1} \det A_{r1} + (-1)^{r+2} a_{r2} \det A_{r2} + \cdots + (-1)^{r+n} a_{rn} \det A_{rn}$$

$$= (-1)^{r+1} b_{s1} \det B_{s1} + (-1)^{r+2} b_{s2} \det B_{s2} + \cdots + (-1)^{r+n} b_{sn} \det B_{sn}$$

$$= -(-1)^{s+1} b_{s1} \det B_{s1} - (-1)^{s+2} b_{s2} \det B_{s2} - \cdots - (-1)^{s+n} b_{sn} \det B_{sn}$$

$$\Rightarrow \det A = -\det B.$$

If B is obtained by interchanging row r and row $s > r + 1$ of A,

$\Rightarrow$ B may be obtained from A by making adjacent row interchanges:

$$\begin{bmatrix} ---\mathbf{a}'_1--- \\ \vdots \\ ---\mathbf{a}'_{r-1}--- \\ ---\mathbf{a}'_r--- \\ ---\mathbf{a}'_{r+1}--- \\ \vdots \\ ---\mathbf{a}'_{s-1}--- \\ ---\mathbf{a}'_s--- \\ ---\mathbf{a}'_{s+1}--- \\ \vdots \\ ---\mathbf{a}'_n--- \end{bmatrix} \xrightarrow[\substack{\text{adjacent row} \\ \text{interchanges}}]{s\text{ - }r} \begin{bmatrix} ---\mathbf{a}'_1--- \\ \vdots \\ ---\mathbf{a}'_{r-1}--- \\ ---\mathbf{a}'_{r+1}--- \\ \vdots \\ ---\mathbf{a}'_{s-1}--- \\ ---\mathbf{a}'_s--- \\ ---\mathbf{a}'_r--- \\ ---\mathbf{a}'_{s+1}--- \\ \vdots \\ ---\mathbf{a}'_n--- \end{bmatrix} \xrightarrow[\substack{\text{adjacent row} \\ \text{interchanges}}]{s\text{ - }r\text{ - }1} \begin{bmatrix} ---\mathbf{a}'_1--- \\ \vdots \\ ---\mathbf{a}'_{r-1}--- \\ ---\mathbf{a}'_s--- \\ ---\mathbf{a}'_{r+1}--- \\ \vdots \\ ---\mathbf{a}'_{s-1}--- \\ ---\mathbf{a}'_r--- \\ ---\mathbf{a}'_{s+1}--- \\ \vdots \\ ---\mathbf{a}'_n--- \end{bmatrix}$$

$$\Rightarrow \det B = (-1)^{s-r}(-1)^{s-r-1} \cdot \det A = (-1)^{2(s-r)-1} \cdot \det A = -\det A$$

Theorem 3.3 (b)

Let A be an $n \times n$ matrix. If B is a matrix obtained by multiplying each entry of some row of A by a scalar k, then $\det B = k \det A$.

Proof (b) If B is obtained by multiplying row r of A by k,

$$\Rightarrow \det B$$

$$= (-1)^{r+1} b_{r1} \det B_{r1} + (-1)^{r+2} b_{r2} \det B_{r2} + \cdots + (-1)^{r+n} b_{rn} \det B_{rn}$$

$$= (-1)^{r+1} ka_{r1} \det A_{r1} + (-1)^{r+2} ka_{r2} \det A_{r2} + \cdots + (-1)^{r+n} ka_{rn} \det A_{rn}$$

$$= k \cdot \det A.$$

Theorem 3.3 (c)

Let A be an $n \times n$ matrix. If B is a matrix obtained by adding a multiple of some row of A to a different row, then $\det B = \det A$.

Proof If $C \in \mathcal{M}_{n\times n}$ has two identical rows, then by (b) det C = - det C, since $C = C$ with the two identical rows interchanged.

$\Rightarrow \det C = 0.$

If B is obtained by adding k times row s of A to row r ($\neq s$),

$$\Rightarrow \det B = (-1)^{r+1} b_{r1} \det B_{r1} + \cdots + (-1)^{r+n} b_{rn} \det B_{rn}$$

$$= (-1)^{r+1}(a_{r1}+ka_{s1})\det A_{r1} + \cdots + (-1)^{r+n}(a_{rn}+ka_{sn})\det A_{rn}$$

$$= (-1)^{r+1} a_{r1} \det A_{r1} + \cdots + (-1)^{r+n} a_{rn} \det A_{rn}$$

$$+ k \cdot [(-1)^{r+1} a_{s1} \det A_{r1} + \cdots + (-1)^{r+n} a_{sn} \det A_{rn}]$$

$= \det A + k \cdot \det C$, where rows r and s of C are identical.

$$\Rightarrow \det B = \det A + k \cdot 0 = \det A$$

Theorem 3.3

Let A be an $n \times n$ matrix.

(a) If B is a matrix obtained by interchanging two rows of A, then $\det B = -\det A$.

(b) If B is a matrix obtained by multiplying each entry of some row of A by a scalar k, then $\det B = k \det A$.

(c) If B is a matrix obtained by adding a multiple of some row of A to a different row, then $\det B = \det A$.

(d) For any $n \times n$ elementary matrix E, we have $\det EA = (\det E)(\det A)$.

Proof (d) If $E \in \mathcal{M}_{n\times n}$ is an elementary matrix obtained by interchanging two rows of I_n, then $\det E = -\det I_n = -1$, and by (a) $\det EA = -\det A = (\det E)(\det A)$. For the other two types of elementary matrices, the proofs are similar.

With the steps 1-4 of the Gaussian elimination algorithm, every $A \in \mathcal{R}^{n \times n}$ may be transformed into a row echelon form by using elementary row operations other than scaling operations.

Matrices in the row echelon form are upper triangular.

If an $n \times n$ matrix A is transformed into an upper triangular matrix U by elementary row operations other than scaling operations, then

$$\det A = (-1)^r u_{11} u_{22} \cdots u_{nn},$$

where r is the number of row interchanges performed.

Proof $E_k \cdots E_2 E_1 A = U \Rightarrow \det(E_k) \cdots \det(E_2)\det(E_1)\det(A) = \det U.$

Since $\det(E_k) = \pm 1$, we have $(-1)^r \det A = \det U$.

Example:

$$A = \begin{bmatrix} 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ -2 & 0 & 4 & -7 \\ 4 & -4 & 4 & 15 \end{bmatrix} \Longrightarrow \begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 0 & 4 & -2 \\ 0 & 1 & 3 & -3 \\ 4 & -4 & 4 & 15 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 0 & 4 & -2 \\ 0 & 1 & 3 & -3 \\ 0 & -4 & 12 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & -4 & 12 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 24 & -11 \end{bmatrix} \Longrightarrow \begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

$$\Rightarrow \det A = (-1)^2 \det U = (-1)^2 \cdot (-2) \cdot 1 \cdot 4 \cdot 1 = -8.$$

Using Gaussian elimination to evaluate determinants is much faster than using cofactor expansion, especially for large matrices.

For any $A \in \mathcal{M}_{n\times n}$, A is not invertible if and only if $\det A = 0$.

Proof rank $A < n \Leftrightarrow$ its row echelon form has the zero bottom row.

Theorem 3.4 (a)(b)

Let A and B be square matrices of the same size. The following statements are true.

(a) A is invertible if and only if $\det A \neq 0$.

(b) $\det AB = (\det A)(\det B)$.

Proof

b) If A is invertible, then $\exists$ elementary matrices $E_1, E_2, \cdots, E_k$, such that $A = E_k \cdots E_2 E_1$.

$\Rightarrow (\det A)(\det B) = (\det E_k) \cdots (\det E_2)(\det E_1)(\det B)$

$= (\det E_k) \cdots (\det E_2)(\det E_1 B) = \cdots$

$= \det(E_k \cdots E_2 E_1 B) = \det AB.$

If A is not invertible, then $\exists$ an invertible P such that $PA = R$, the reduced row echelon form of A.

$\Rightarrow R$, and thus RB, have the zero bottom rows.

$\Rightarrow (\det P)(\det AB) = \det P(AB) = \det RB = 0 \Rightarrow \det AB = 0$,

but $(\det A)(\det B) = 0 \cdot (\det B) = 0$.

Theorem 3.4 (c)

Let A be a square matrix. Then $\det A^T = \det A$.

Proof

(c) If A is invertible, then $\exists$ elementary matrices $E_1, E_2, \cdots, E_k$, such that $A = E_k \cdots E_2 E_1$, and $A^T = E_1^T E_2^T \cdots E_k^T$.

$\Rightarrow \det A^T = \det(E_1^T E_2^T \cdots E_k^T) = (\det E_1^T)(\det E_2^T) \cdots (\det E_k^T)$

$= (\det E_1)(\det E_2) \cdots (\det E_k)$

$= (\det E_k) \cdots (\det E_2)(\det E_1)$

$= \det (E_k \cdots E_2 E_1) = \det A.$

If A is not invertible, then A^T is not invertible, otherwise $(A^T)^T = A$ would be invertible.

$\Rightarrow \det A^T = 0 = \det A.$

Note: $\det E_i^T = \det E_i$

Theorem 3.4 (d)

If A is an intertible matrix. Then $\det A^{-1} = \frac{1}{\det A}$.

Proof

Theorem 3.4

Let A and B be square matrices of the same size. The following statements are true.

(a) A is invertible if and only if $\det A \neq 0$.

(b) $\det AB = (\det A)(\det B)$.

(c) $\det A^T = \det A$.

(d) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.

Example:

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & c \\ 2 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & c+2 \\ 2 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & c+2 \\ 0 & 3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & c+2 \\ 0 & 0 & 3c+9 \end{bmatrix}$$

$\therefore$ The matrix is invertible if and only if $c \neq -3$.

Example:

$$M = \begin{array}{cc} \left[\begin{array}{cc} A & B \\ O & C \end{array}\right] & \begin{array}{l} \}m \\ \}n \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{m} & \underbrace{\quad}_{n} \end{array} & \end{array} = \begin{bmatrix} I_m & \overset{m\times n}{O'} \\ O & C \end{bmatrix} \begin{bmatrix} A & B \\ O & I_n \end{bmatrix}$$

$$\Rightarrow \det M = \det \begin{bmatrix} I_m & O' \\ O & C \end{bmatrix} \cdot \det \begin{bmatrix} A & B \\ O & I_n \end{bmatrix} = (\det C)(\det A).$$

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \det(A + B) \neq \det A + \det B$$

Homework Set for Section 3.2

Section 3.2: Problems 7, 13, 22, 36, 39-44