CHAPTER 3
DETERMINANTS

- The determinant of a **square matrix** is a **scalar** the provides information about the matrix.
  - Ex: **Invertibility** of the matrix.
- In this chapter, we will learn
  - How to calculate the determinant of a matrix.
  - Properties of determinants
- In this course, we will use determinants will be used in Chapter 5 when we learn to calculate the **eigenvalues** of a square matrix (Chapter 5).

Definition. (Submatrix for cofactor)

Suppose $A = [a_{ij}] \in \mathcal{M}_{n\times n}$ is an $n \times n$ square matrix.
An $(n-1) \times (n-1)$ matrix A_{ij} is defined as the submatrix A obtained by removing the ith row and the jth column of A.

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

ith row

jth column

Definition. (Determinants and Cofactors)

Suppose $A = [a_{ij}] \in \mathcal{M}_{n\times n}$ is an $n \times n$ square matrix.
The **determinant** of A, denoted by $\det A$ or $|A|$, is defined as $\det A = a_{11}$ for $n = 1$ and

$$\det A = a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

for $n > 1$. The (i,j)**-cofactor** c_{ij} of A is defined as $(-1)^{i+j} \det A_{ij}$.

Example:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A_{11} = [d], \text{ and } A_{12} = [c] \Rightarrow \det A = a \cdot \det A_{11} - b \cdot \det A_{12} = ad - bc$$

A is not invertible $\Leftrightarrow [\ a \ \ c\ ]^T$ and $[\ b \ \ d\ ]^T$ are L.D.

$\Leftrightarrow a = c = 0$ or $[\ a \ \ c\ ]^T = k[\ b \ \ d\ ]^T$ for some k

$\Leftrightarrow \det A = ad - bc = 0$

Later it will be shown that for any $A \in \mathcal{M}_{n\times n}$, A is not invertible if and only if $\det A = 0$.

Example: Find the scalars c for which A - cI_2 is not invertible, where

$$A = \begin{bmatrix} 11 & 12 \\ -8 & -9 \end{bmatrix}$$

Solution:

$$\det(A - cI_2) = \det \begin{bmatrix} 11 - c & 12 \\ -8 & -9 - c \end{bmatrix}$$

$\Rightarrow$ for c = -1 and 3, A - cI_2 is not invertible.

Example (any 3 x 3 matrix):

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

$$= (-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$\Rightarrow$ there are six terms, three with "+" sign, and three with "-" sign.

Cofactor expansion along different rows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

Cofactor expansion along the 1st row

$$= (-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23}$$

Cofactor expansion along the 2nd row

$$= (-1)^{2+1}a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2}a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+3}a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} + a_{23}a_{12}a_{31} - a_{23}a_{11}a_{32}$$

$$a_{31}c_{31} + a_{32}c_{32} + a_{33}c_{33}$$

Cofactor expansion along the 3rd row

$$= (-1)^{3+1}a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + (-1)^{3+2}a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + (-1)^{3+3}a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Observation:
For any 3 x 3 matrices, the cofactor expansions along the 1^{st}, 2^{nd}, or 3^{rd} rows are all the same.

Question:
Can we argue that, for any n x n matrices (n is any positive integer), the cofactor expansion along the ith row is also a constant for i=1, 2, …, n?

Theorem 3.1 (**Cofactor expansion** of A along row i)

For any $i = 1, 2, ..., n$, we have

$$\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in},$$

where c_{ij} denotes the (i, j)-cofactor of A.

Proof By induction on the size k of the matrix A, for $k = 1, 2, 3$ prove by brutal force.

Assume for $k = n - 1$ the Theorem holds. We will show that for $k = n$, the Theorem also holds.

For $k = n$, $\color{red}{\det A} = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$

$$= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n},$$

$$= a_{11} \begin{vmatrix} \quad \end{vmatrix} + a_{12} \begin{vmatrix} \quad \end{vmatrix} + \cdots + a_{1n} \begin{vmatrix} \quad \end{vmatrix}$$

Signs are neglected.

$$= a_{11} \left(a_{i2} \begin{vmatrix} \quad \end{vmatrix} + a_{i3} \begin{vmatrix} \quad \end{vmatrix} + \cdots + a_{in} \begin{vmatrix} \quad \end{vmatrix} \right)$$

$$+ a_{12} \left(a_{i1} \begin{vmatrix} \quad \end{vmatrix} + a_{i3} \begin{vmatrix} \quad \end{vmatrix} + \cdots + a_{in} \begin{vmatrix} \quad \end{vmatrix} \right)$$

$$+ \cdots$$

$$+ a_{1n} \left(a_{i1} \begin{vmatrix} \quad \end{vmatrix} + a_{i2} \begin{vmatrix} \quad \end{vmatrix} + \cdots + a_{i,n-1} \begin{vmatrix} \quad \end{vmatrix} \right)$$

$$= a_{i1}\left(a_{12}\begin{vmatrix} \quad \end{vmatrix} + a_{13}\begin{vmatrix} \quad \end{vmatrix} + \cdots + a_{1n}\begin{vmatrix} \quad \end{vmatrix}\right)$$

$$+a_{i2}\left(a_{11}\begin{vmatrix} \quad \end{vmatrix} + a_{13}\begin{vmatrix} \quad \end{vmatrix} + \cdots + a_{1n}\begin{vmatrix} \quad \end{vmatrix}\right)$$

$$+\cdots$$

$$+a_{in}\left(a_{11}\begin{vmatrix} \quad \end{vmatrix} + a_{12}\begin{vmatrix} \quad \end{vmatrix} + \cdots + a_{1,n-1}\begin{vmatrix} \quad \end{vmatrix}\right)$$

$$= a_{i1}\begin{vmatrix} \quad \end{vmatrix} + a_{i2}\begin{vmatrix} \quad \end{vmatrix} + \cdots + a_{in}\begin{vmatrix} \quad \end{vmatrix}$$

Example:

Consider the matrix $M = \begin{bmatrix} 1 & 2 & 3 & 8 & 5 \\ 4 & 5 & 6 & 9 & 1 \\ 7 & 9 & 8 & 4 & 7 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Find $\det M$.

$$\det M = 0+0+0+0+1(-1)^{5+5}\det M_{55} = \det M_{55} = \det \begin{bmatrix} 1 & 2 & 3 & 8 \\ 4 & 5 & 6 & 9 \\ 7 & 9 & 8 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 0+0+0+1(-1)^{4+4}\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Generalizing the idea, we have for any $A \in \mathcal{R}^{m\times m}$ and $B \in \mathcal{R}^{m\times n}$

$$\det \begin{bmatrix} A & B \\ O & I_n \end{bmatrix} = \det A.$$

Question: For an n x n matrix A in general, how many multiplications and how many additions do you need to calculate det A?

Computational Complexity

Without zero entries, the cofactor expansion of an arbitrary $n \times n$ matrix requires at least $n!$ arithmetic operations.

For $n = 20$, $n! > 2.433 \times 10^{18}$.
For $n = 100$, $n! > 9.333 \times 10^{157}$.

We need to come up with something faster. For example:

$$\det\begin{bmatrix} 3 & -4 & -7 & -5 \\ 0 & 8 & -2 & 6 \\ 0 & 0 & 9 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4(-1)^{4+4} \cdot \det\begin{bmatrix} 3 & -4 & -7 \\ 0 & 8 & -2 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= 4 \cdot 9(-1)^{3+3} \cdot \det\begin{bmatrix} 3 & -4 \\ 0 & 8 \end{bmatrix}$$

$$= 4 \cdot 9 \cdot 8(-1)^{2+2} \cdot \det\begin{bmatrix} 3 \end{bmatrix} = 4 \cdot 9 \cdot 8 \cdot 3.$$

upper triangular matrix

Definitions

An $n \times n$ matrix A is said to be **lower triangular** if $a_{ij} = 0$ for all i, j that satisfy $i < j$.
An $n \times n$ matrix A is said to be **upper triangular** if $a_{ij} = 0$ for all i, j that satisfy $i > j$.

Theorem 3.2

The determinant of an upper triangular $n \times n$ matrix or a lower triangular $n \times n$ matrix equals the product of its diagonal entries.

Corollaries

(a) $\det I_n = 1$

(b) For any upper or lower triangular $A \in \mathcal{R}^{m \times m}$,

$$\det A = 0 \Leftrightarrow \text{ at least one diagonal entry of } A \text{ is zero.}$$

Homework Set for Section 3.1

Section 3.1: Problems 1, 2, 9, 12, 14, 15, 37, 44