2.8 Composition and Invertibility of Linear Transformations

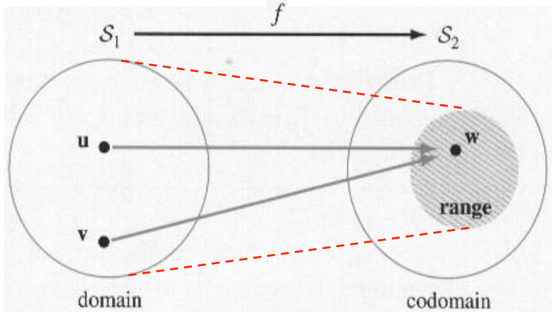
The standard matrix of a linear transformation *T* can be used to find a **generating set** for **the range of *T***.

Example: $T : \mathcal{R}^3 \to \mathcal{R}^2$,

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 4x_2 \\ 2x_1 + x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Now, $\mathbf{w} \in \text{range}(T) \Leftrightarrow \mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in \mathcal{R}^2$.



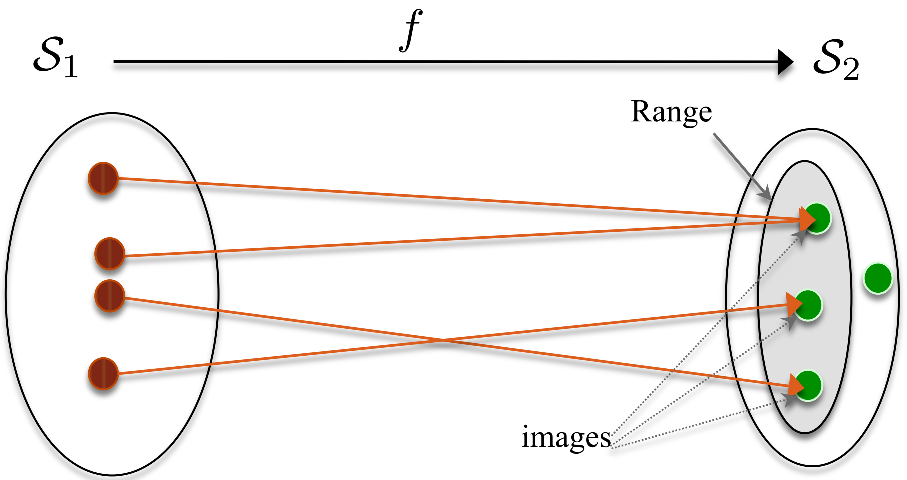
Property

The range of a linear transformation equals the span of the columns of its standard matrix.

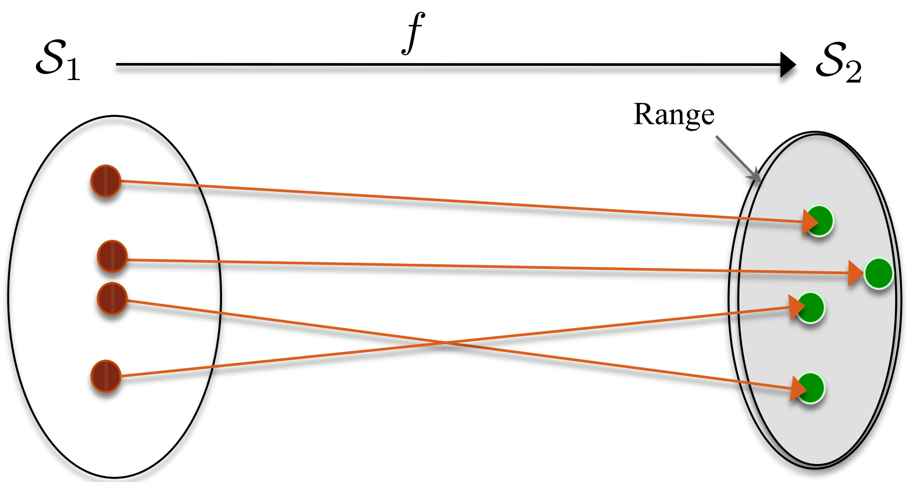
Proof

Definition

A function $f : \mathcal{R}^n \to \mathcal{R}^m$ is said to be **onto** if its range is all of $\mathcal{R}^m$; that is, if every vector in $\mathcal{R}^m$ is an image.



not onto



onto

Definition

A function $f : \mathcal{R}^n \to \mathcal{R}^m$ is said to be **onto** if its range is all of $\mathcal{R}^m$; that is, if every vector in $\mathcal{R}^m$ is an image.

(Review) Theorem 1.6

The following statements about an $m \times n$ matrix A are equivalent:
(a) The span of the columns of A is $\mathcal{R}^m$.
(c) the rank of A is m.

Theorem 2.10

Let $T : \mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation with standard matrix A. The following conditions are equivalent:
(a) T is onto; that is, the range of T is $\mathcal{R}^m$.
(b) The columns of A form a generating set for $\mathcal{R}^m$.
(c) $\text{rank}(A) = m$.

Proof

Example: Is T onto?

$$T : \mathcal{R}^3 \to \mathcal{R}^3 \text{ with } T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ x_1 + 3x_2 + 6x_3 \\ 2x_1 + 5x_2 + 10x_3 \end{bmatrix}$$

$\Rightarrow A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$ with the reduced row echelon form

$$R = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

The system of linear equations $A\mathbf{x} = \mathbf{b}$ may be written as $T_A(\mathbf{x}) = \mathbf{b}$.
$A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the range of T_A.
T_A is not onto if and only if $\exists$ $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent.

Definition

A function $f : \mathcal{R}^n \to \mathcal{R}^m$ is said to be **onto** if its range is all of $\mathcal{R}^m$; that is, if every vector in $\mathcal{R}^m$ is an image.

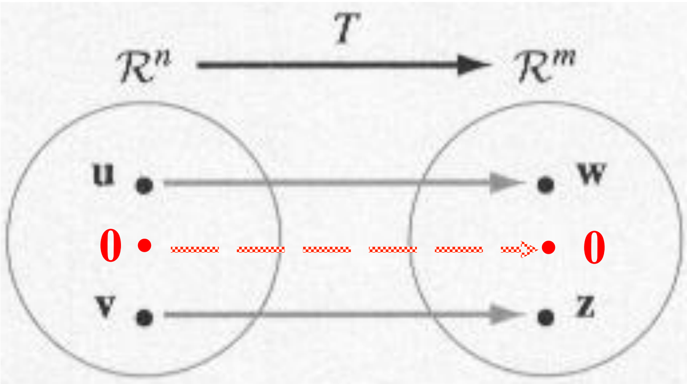
(Review) Theorem 1.6

The following statements about an $m \times n$ matrix A are equivalent:
(a) The span of the columns of A is $\mathcal{R}^m$.
(c) the rank of A is m.

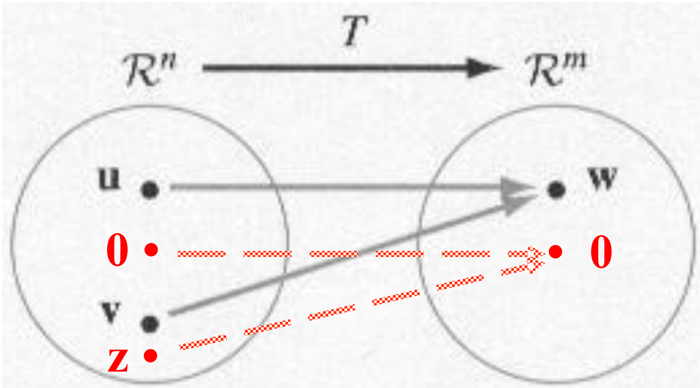
Proof

Definition

A function $f : \mathcal{R}^n \to \mathcal{R}^m$ is said to be **one-to-one** if every pair of distinct vectors in $\mathcal{R}^n$ has distinct images. That is, if $\mathbf{u}$ and $\mathbf{v}$ are distinct vectors in $\mathcal{R}^n$, then $f(\mathbf{u})$ and $f(\mathbf{v})$ are distinct vectors in $\mathcal{R}^m$.



one-to-one

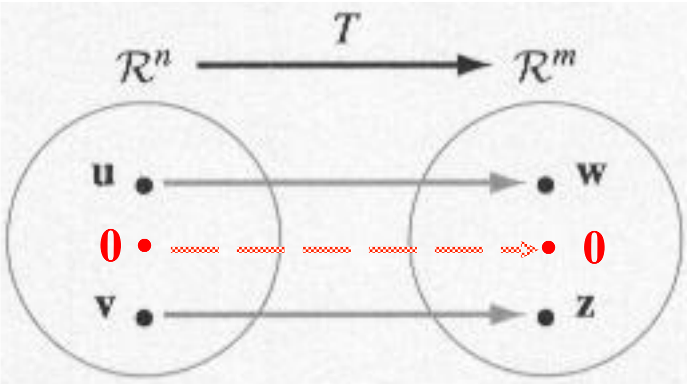


not one-to-one

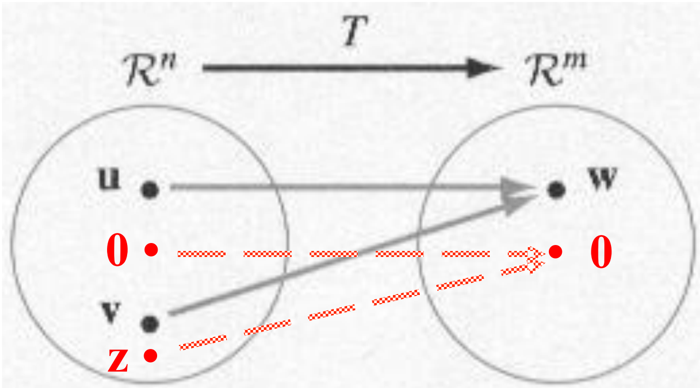
An equivalent condition for a function f to be one-to-one is that $f(\mathbf{u}) = f(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.

Definition

A function $f : \mathcal{R}^n \to \mathcal{R}^m$ is said to be **one-to-one** if every pair of distinct vectors in $\mathcal{R}^n$ has distinct images. That is, if $\mathbf{u}$ and $\mathbf{v}$ are distinct vectors in $\mathcal{R}^n$, then $f(\mathbf{u})$ and $f(\mathbf{v})$ are distinct vectors in $\mathcal{R}^m$.



one-to-one



not one-to-one

Definition

Let $T : \mathcal{R}^n \to \mathcal{R}^m$ be linear. The **null space** of T is the set of all $\mathbf{v} \in \mathcal{R}^n$ such that $T(\mathbf{v}) = \mathbf{0}$.

Property

A linear transformation is one-to-one if and only if its null space contains only $\mathbf{0}$.

Example:

$$T : \mathcal{R}^3 \rightarrow \mathcal{R}^2 \text{ with } T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 + x_2 - 3x_3 \end{bmatrix}$$

Find a generating set for the null space of T.

the null space of T is the set of solutions to $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} \quad \end{bmatrix}$$

is the standard matrix of T, and has the reduced row echelon form

$$R = \begin{bmatrix} \quad \end{bmatrix}$$

The reduced row echelon form corresponds to the linear equations

$$\begin{aligned} x_1 - x_2 \qquad &= 0 \\ x_3 &= 0 \end{aligned}$$

so a generating set for the null space of T is $\{ [\ 1 \ \ 1 \ \ 0\ ]^T \}$

Theorem 2.11

Let $T : \mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation with standard matrix A. The following conditions are equivalent:

(a) T is one-to-one.

(b) The null space of T consists only of the zero vector.

(c) The columns of A are linearly independent.

(d) $\text{rank}(A) = n$.

(Review) Theorem 1.8

The following statements about an $m \times n$ matrix A are equivalent:

(a) The columns of A are linearly independent.

(d) $\text{rank}(A) = n$.

Proof

Example:

$$T : \mathcal{R}^3 \to \mathcal{R}^3 \text{ with } T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ x_1 + 3x_2 + 6x_3 \\ 2x_1 + 5x_2 + 10x_3 \end{bmatrix} = A\mathbf{x}$$

The standard matrix A has rank 2 (nullity 1), so T is not one-to-one.

If T_A is one-to-one and the solution of $A\mathbf{x} = \mathbf{b}$ exists, then the solution is unique. Conversely, if there is at most one solution to $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b}$, then T_A is one-to-one.

Example:

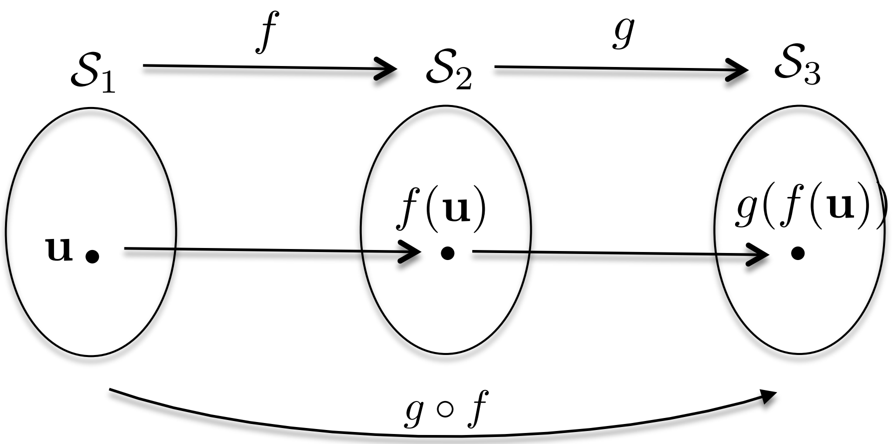
$$A = \begin{bmatrix} 0 & 0 & 1 & 3 & 3 \\ 2 & 3 & 1 & 5 & 2 \\ 4 & 6 & 1 & 6 & 2 \\ 4 & 6 & 1 & 7 & 1 \end{bmatrix} \xrightarrow[\text{echelon form}]{\text{row reduced}} R = \begin{bmatrix} 1 & 1.5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $A = 3$ and nullity $A = 2$. $\Rightarrow$ T_A is not onto and not one-to-one.

$\Rightarrow$ $A\mathbf{x} = \mathbf{b}$ is inconsistent for $\mathbf{b}$ not in the range of T_A, and the solutions of $A\mathbf{x} = \mathbf{b}$ is never unique.

Definition.

Given two functions $f : S_1 \to S_2$ and $g : S_2 \to S_3$, the **composite** $g \circ f : S_1 \to S_3$ is the function defined by $(g \circ f)(\mathbf{u}) = g(f(\mathbf{u}))$ for all $\mathbf{u}$ in S_1. Sometimes the "$\circ$" is dropped for brevity.



Example: $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_1 + x_2 \end{bmatrix}$ $g\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_3 \\ 3x_2 \end{bmatrix}$

$$\Rightarrow \qquad (g \circ f)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 - (x_1 + x_2) \\ 3x_1 x_2 \end{bmatrix}$$

Given $A \in \mathcal{R}^{m \times n}$ and $B \in \mathcal{R}^{p \times m}$, we have $T_{BA}(\mathbf{v}) = (BA)\mathbf{v} = B(A\mathbf{v})$ $= B(T_A(\mathbf{v})) = T_B(T_A(\mathbf{v}))$ for all $\mathbf{v} \in \mathcal{R}^n$. Thus $T_{BA} = T_B \circ T_A = T_B T_A$.

Theorem 2.12

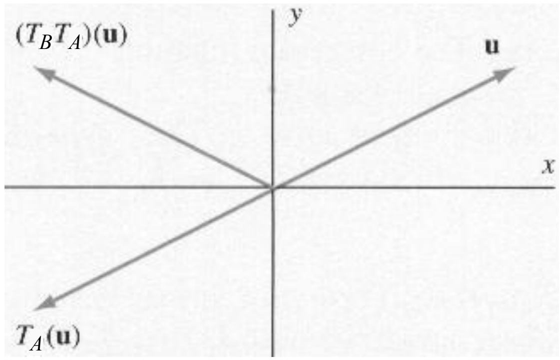
If $T : \mathcal{R}^n \to \mathcal{R}^m$ and $U : \mathcal{R}^m \to \mathcal{R}^p$ are linear transformations with standard matrices A and B, respectively, then the composition $UT : \mathcal{R}^n \to \mathcal{R}^p$ is also linear, and its standard matrix is BA.

Proof

Example: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow T_A$: rotation by 180° in $\mathbb{R}^2$.

$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow T_B$: reflection about the x-axis in $\mathbb{R}^2$.

$$T_{BA}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = (BA)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix}$$



$\Rightarrow T_{BA}$: reflection about the y-axis in $\mathbb{R}^2$
= rotation by 180° followed by
reflection about the x-axis in $\mathbb{R}^2$.

Definition.
A function $f : S_1 \to S_2$ is **invertible** if there exists a function $g : S_2 \to S_1$ such that $(g \circ f)(\mathbf{v}) = g(f(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v}$ in S_1 and $(f \circ g)(\mathbf{u}) = f(g(\mathbf{u})) = \mathbf{u}$ for all $\mathbf{u}$ in S_2. If f is invertible, then the function g is unique, called the **inverse** of f and denoted as f^{-1}.

If $A \in \mathcal{R}^{n\times n}$ is invertible, then for all $\mathbf{v} \in \mathbb{R}^n$ we have $T_A(T_{A^{-1}}(\mathbf{v})) = (T_A T_{A^{-1}})(\mathbf{v}) = T_{AA^{-1}}(\mathbf{v}) = T_{I_n}(\mathbf{v}) = I_n\mathbf{v} = \mathbf{v}$, and $T_{A^{-1}}(T_A(\mathbf{v})) = \mathbf{v}$. Thus $T_A^{-1} = T_{A^{-1}}$.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 2v_2 \\ 3v_1 + 5v_2 \end{bmatrix}$$

$$T_A^{-1}\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T_{A^{-1}}\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -5v_1 + 2v_2 \\ 3v_1 - v_2 \end{bmatrix}$$

Theorem 2.13

Let $T : \mathcal{R}^n \to \mathcal{R}^n$ be a linear transformation with standard matrix A. Then T is invertible if and only if A is invertible, in which case $T^{-1} = T_{A^{-1}}$. Thus T^{-1} is linear, and its standard matrix is A^{-1}.

Proof

Property of T	The number of solutions of $A\mathbf{x}=\mathbf{b}$	Property of the columns of A	Property of the rank of A	Theorem
T is onto	$A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ in $\mathcal{R}^m$.	The columns of A are a generating set for $\mathcal{R}^m$.	rank $A = m$	**2.10**
T is one-to-one	$A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$ in $\mathcal{R}^m$.	The columns of A are linearly independent.	rank $A = n$	**2.11**
T is invertible	$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathcal{R}^m$	The columns of A are a linearly independent generating set for $\mathcal{R}^m$.	rank $A = m = n$	**2.13**

Homework Set for Section 2.8

Section 2.8: Problems 1, 5, 13, 17, 19, 25, 29, 33, 37