Theorem 2.5

Let A be an $n \times n$ matrix. Then A is invertible if and only if the reduced row echelon form of A is I_n.

Proof First, suppose that A is invertible.

$$\text{For } \mathbf{v} \in \mathcal{R}^n,\ A\mathbf{v} = \mathbf{0} \Rightarrow A^{-1}A\mathbf{v} = A^{-1}\mathbf{0} = \mathbf{0} \Rightarrow \mathbf{v} = \mathbf{0}$$
$$\Rightarrow \text{rank } A = n \text{ by Theorem 1.8 (d)(f)}$$
$$\Rightarrow \text{reduced row echelon form of } A \text{ is } I_n \text{ since } A \text{ is } n \times n$$

So, the reduced row echelon form of A must equal I_n.

Theorem 2.5

Let A be an $n \times n$ matrix. Then A is invertible if and only if the reduced row echelon form of A is I_n.

Proof Conversely, suppose that the reduced row echelon form of A equals I_n.

Therefore, A is invertible.

Theorem 2.5

Let A be an $n \times n$ matrix. Then A is invertible if and only if the reduced row echelon form of A is I_n.

Examples:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix}$$

has the reduced row echelon form $I_n \Rightarrow$ invertible.

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ not invertible.

Algorithm for Matrix Inversion

Let A be an $n \times n$ matrix. Use elementary row operations to transform $\begin{bmatrix} A & I_n \end{bmatrix}$ into the form $\begin{bmatrix} R & B \end{bmatrix}$, where R is a matrix in reduced row echelon form. Then either

(a) $R = I_n$, in which case A is invertible and $B = A^{-1}$; or

(b) $R \neq I_n$, in which case A is not invertible.

Proof

(a) $[R\ B] = P[A\ I_n] = [PA\ PI_n] = [PA\ P]$

(b) By Thm 2.5

Algorithm for Matrix Inversion

Let A be an $n \times n$ matrix. Use elementary row operations to transform $\begin{bmatrix} A & I_n \end{bmatrix}$ into the form $\begin{bmatrix} R & B \end{bmatrix}$, where R is a matrix in reduced row echelon form. Then either

(a) $R = I_n$, in which case A is invertible and $B = A^{-1}$; or

(b) $R \neq I_n$, in which case A is not invertible.

Example:

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 4 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & -7 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right] \qquad A^{-1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -20 & 6 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & 4 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right] = \begin{bmatrix} I_3 & B \end{bmatrix}$$

Algorithm for Computing $A^{-1}B$

Let A be an invertible $n \times n$ matrix and B be an $n \times p$ matrix. Suppose that the $n \times (n+p)$ matrix $\begin{bmatrix} A & B \end{bmatrix}$ is transformed by means of elementary row operations into the matrix $\begin{bmatrix} I_n & C \end{bmatrix}$ in reduced row echelon form. Then $C = A^{-1}B$.

Proof $\begin{bmatrix} I_n & C \end{bmatrix} = P\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} PA & PB \end{bmatrix}$

Example:

$$\begin{bmatrix} A & B \end{bmatrix} = \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 2 & -1 \\ 2 & 5 & 1 & 1 & 3 \\ 2 & 4 & 1 & 0 & 2 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 2 & -1 \\ 0 & 1 & -1 & -3 & 5 \\ 0 & 0 & -1 & -4 & 4 \end{array}\right]$$

Clearly, it is easier to compute $A^{-1}B$ directly than to compute A^{-1} first and get $A^{-1}B$ afterwards.

$$\to \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 2 & -1 \\ 0 & 1 & -1 & -3 & 5 \\ 0 & 0 & 1 & 4 & -4 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 2 & 0 & -2 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 & -4 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -4 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 & -4 \end{array}\right]$$

$A^{-1}B$

Theorem 2.6 (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. The following statments are equivalent:

(a) A is invertible.

(b) The reduced row echelon form of A is I_n.

(c) The rank of A equals n.

(d) The span of the columns of A is $\mathcal{R}^n$.

(e) The equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathcal{R}^n$.

(f) The nullity of A equals zero.

(g) The columns of A are linearly independent.

(h) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.

(i) There exists an $n \times n$ matrix B such that $BA = I_n$

(j) There exists an $n \times n$ matrix C such that $AC = I_n$

(k) A is a product of elementary matrices.

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Proof

(a) ⇔ (b): Theorem 2.5.

(d) ⇔(e): By definition of span and matrix-vector products.

(b) ⇔ (c): rank(A) = the number of nonzero rows of reduced row echelon form of A, $I_n = n$.

(a) ➔ (e) $\mathbf{x} = A^{-1}\mathbf{b}$.

(e) ➔ (a) $A\mathbf{x} = \mathbf{b}$ has at least a solution for any $\mathbf{b}$ in R^n.

Let $A = PR$, where R is the reduced row echelon form.

Suppose A is NOT invertible, then R is not I_n, and the last row of R is zero.

Theorem 2.6 (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. The following statments are equivalent:

(a) A is invertible.

(b) The reduced row echelon form of A is I_n.

(c) The rank of A equals n.

(f) The nullity of A equals zero.

(g) The columns of A are linearly independent.

Proof

(c) ⇔ (f) Nullity = n – rank(A) = n- n = 0. (# of free variables = 0)

(g) ⇔ (c) Thm 1.8 (a)(d).

(g) ⇔ (f) Thm 1.8 (a)(c).

(g) ⇔ The columns of R are linearly independent. They are all pivot columns.⇔R=In ⇔(b).

Theorem 2.6 (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. The following statments are equivalent:

(a) A is invertible.

(b) The reduced row echelon form of A is I_n.

(g) The columns of A are linearly independent.

(h) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.

(k) A is a product of elementary matrices.

Proof

(a) ➔ (h): $\mathbf{x} = A^{-1}\,\mathbf{0} = \mathbf{0}$;

(h) ⇔ (g): By definitions of l.i. and matrix-vector product.

(b) ➔ (k) $I_n = E_k \ldots E_2\, E_1\, A$ ➔ $E_k^{-1} = E_{k-1} \ldots E_2 E_1 A$ ➔ $A = E_1^{-1}\, E_2^{-1} \ldots E_k^{-1}$

(k) ➔ (a) $A =$ is invertible (Thm 2.2 b)

Theorem 2.6 (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. The following statments are equivalent:

(a) A is invertible.

(b) The reduced row echelon form of A is I_n.

(e) The equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathcal{R}^n$.

(h) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.

(i) There exists an $n \times n$ matrix B such that $BA = I_n$

(j) There exists an $n \times n$ matrix C such that $AC = I_n$

Proof (a) ➔ (i) obvious. (a)➔(j) obvious (by def.)

(i) ➔ (h) ➔ (a) : Let $\mathbf{x}$ be any vector in $\mathcal{R}^n$ such that $A\mathbf{x} = \mathbf{0}$.

Then $\mathbf{x} =$

(j) ➔ (e) ➔ (a): Let $\mathbf{b}$ be any vector in $\mathcal{R}^n$ and let $\mathbf{v} = C\mathbf{b}$.

Then $A\mathbf{v} =$

Theorem 2.6 (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. The following statments are equivalent:

(a) A is invertible.

(b) The reduced row echelon form of A is I_n. $R = I_n$

(c) The rank of A equals n. $\operatorname{rank}(A) = n$

(d) The span of the columns of A is $\mathcal{R}^n$. $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\} = \mathcal{R}^n$

(e) The equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathcal{R}^n$.

$\forall \mathbf{b} \in \mathcal{R}^n, \exists \mathbf{x} \in \mathcal{R}^n s.t. A\mathbf{x} = \mathbf{b}$

(f) The nullity of A equals zero. $\operatorname{nullity}(A) = 0$

(g) The columns of A are linearly independent. $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ is l.i.

(h) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.

(i) There exists an $n \times n$ matrix B such that $BA = I_n$

(j) There exists an $n \times n$ matrix C such that $AC = I_n$

(k) A is a product of elementary matrices. $A = E_1 E_2 \cdots E_k$

Note that though only one of $BA = I_n$ or $AC = I_n$ is needed to show the invertibility of A, the matrix A has to be square.

There are nonsqaure matrices A and C for which the product AC is an identity matrix. For instance, let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1 \\ -1 & -1 \\ 0 & 2 \end{bmatrix}.$$

Then $AC =$

Of course, A and C are not invertible.

Homework Set for Section 2.4

Section 2.4: Problems 1, 8, 14, 19, 21, 22, 27, 28, 29, 31, 32.