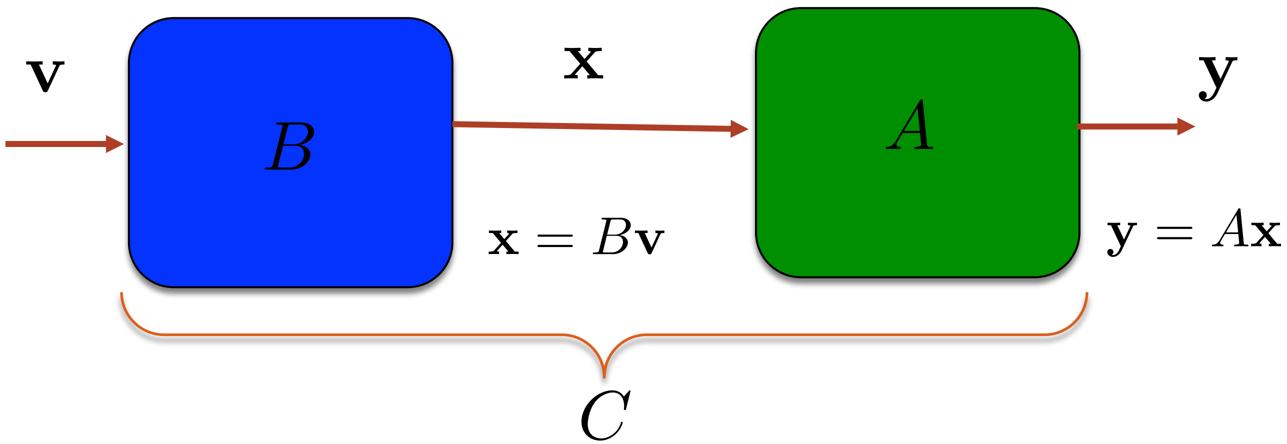
CHAPTER 2
MATRICES AND LINEAR TRANSFORMATIONS

2.1 Matrix Multiplication

From **matrix-vector products** to **matrix-matrix multiplication**.

Let $\mathbf{v}, \mathbf{x}, \mathbf{y} \in \mathcal{R}^n$. Suppose A and B are $n \times n$ matrices.



Does there exist an n x n matrix C such that $\mathbf{y} = C\mathbf{v}$? (for all $\mathbf{v} \in \mathcal{R}^n$)

In general, given $A \in \mathcal{R}^{m \times n}$ and $B \in \mathcal{R}^{n \times p}$, we wish to find $C \in \mathcal{R}^{m \times p}$ such that $C\mathbf{v} = A(B\mathbf{v})$ for all $\mathbf{v} \in \mathcal{R}^p$.

Proposition

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then there is a unique $m \times p$ matrix C such that $C\mathbf{v} = A(B\mathbf{v})$ for every $p \times 1$ vector $\mathbf{v}$. Furthermore,

$$C = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

Proof

Existence:

Uniqueness:

Proposition

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then there is a unique $m \times p$ matrix C such that $C\mathbf{v} = A(B\mathbf{v})$ for every $p \times 1$ vector $\mathbf{v}$. Furthermore,

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Proof

Existence:

$$\begin{aligned} A(B\mathbf{v}) &= A(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_p\mathbf{b}_p) && \text{Def. of Matrix-Vector products} \\ &= A(v_1\mathbf{b}_1) + A(v_2\mathbf{b}_2) + \cdots + A(v_p\mathbf{b}_p) && \text{Theorem 1.3(a)} \\ &= v_1 A\mathbf{b}_1 + v_2 A\mathbf{b}_2 + \cdots + v_p A\mathbf{b}_p && \text{Theorem 1.3(b)} \\ &= \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix} \mathbf{v}. && \text{Def. of Matrix-Vector products} \end{aligned}$$

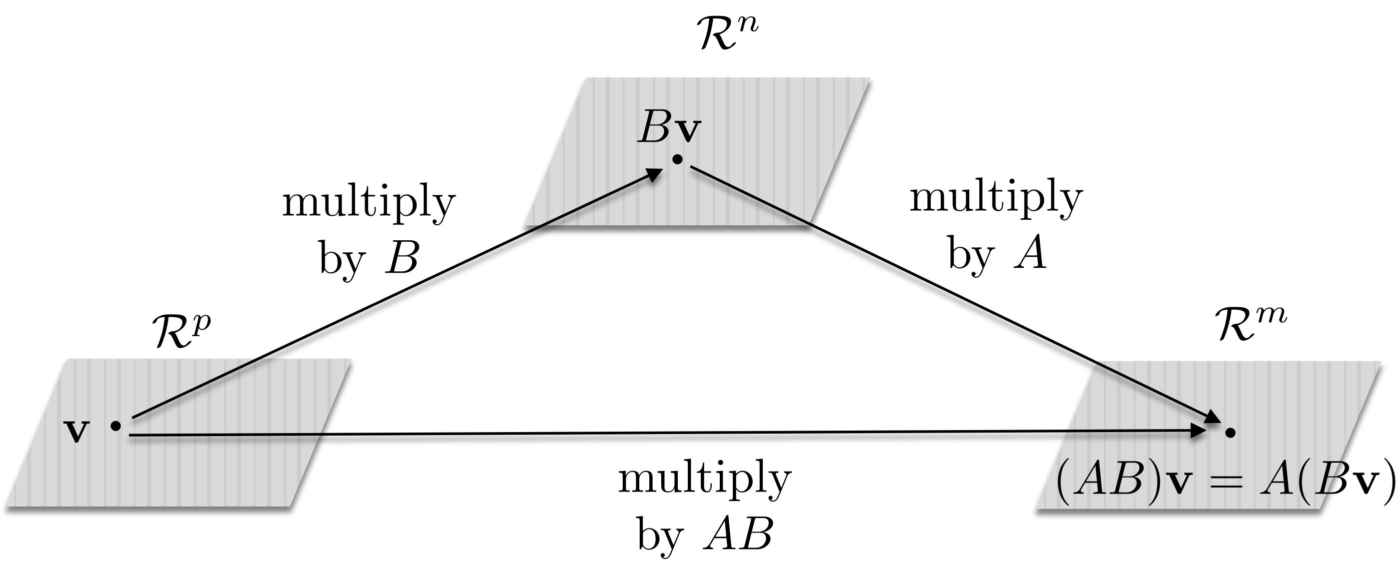
Uniqueness: Suppose there is another $Q \in \mathcal{R}^{m \times p}$ such that $Q\mathbf{v} = A(B\mathbf{v})$ for all $\mathbf{v} \in \mathcal{R}^p$. Then $\mathbf{q}_j = Q\mathbf{e}_j = A(B\mathbf{e}_j) = A\mathbf{b}_j = \mathbf{c}_j$, i.e., $Q = C$.

Definition

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the **(matrix) product** AB to be the $m \times p$ matrix whose jth column is $A\mathbf{b}_j$. That is,

$$AB = \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{array}\right].$$

Dimension requirement: $(m \times n)(n \times p) = (m \times p)$



The Row-Column Rule for the (i, j)-entry of a Matrix Product

$$\begin{array}{r} \\ \\ \text{row } i \text{ of } A \longrightarrow \\ \\ \\ \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{array}{c} \begin{array}{ccccc} & & \text{column } j \text{ of } B & & \\ & & \downarrow & & \end{array} \\ \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} \end{array}$$

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Property

For any $m \times n$ matrix A, any $n \times p$ matrix B, and any $p \times 1$ vector $\mathbf{v}$,

$$(AB)\mathbf{v} = A(B\mathbf{v}).$$

Matrix Multiplication Is NOT Commutative

For arbitrary matrices A and B, AB need not equal BA.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Questions:

1) Does there exist a matrix K such that $KA = A$ for all $A \in \mathcal{M}_{m\times n}$?

2) If $C = AB$, then how can we express C^T in terms of A and B?

Theorem 2.1

Let A and B be $k \times m$ matrices, C be an $m \times n$ matrix, and P and Q be $n \times p$ matrices. Then the following statements are true:

(a) $s(AC) = (sA)C = A(sC)$ for any scalar s.
(b) $A(CP) = (AC)P$. (associative law of matrix multiplication)
(c) $(A + B)C = AC + BC$. (right distributive law)
(d) $C(P + Q) = CP + CQ$. (left distributive law)
(e) $I_k A = A = AI_m$.
(f) The product of any matrix and a zero matrix is a zero matrix.
(g) $(AC)^T = C^T A^T$.

Proof You show (a), (d), (e), and (f).

(b) dimension check $(AC)P$: $[(k\times m)(m\times n)](n\times p) = (k\times p)$

$A(CP)$: $(k\times m)[(m\times n)(n\times p)] = (k\times p)$;

Let $\mathbf{u}_j \equiv$ column j of $CP = C\mathbf{p}_j \Rightarrow$ column j of $A(CP) = A\mathbf{u}_j = A(C\mathbf{p}_j)$,

also, column j of $(AC)P = (AC)\mathbf{p}_j = A(C\mathbf{p}_j)$ by definition.

(c) you do the dimension check;

column j of $(A + B)C = (A + B)\mathbf{c}_j = A\mathbf{c}_j + B\mathbf{c}_j$

$= \text{column } j \text{ of } AC + BC$

(g) you do the dimension check;

the (i,j)-entry of $(AC)^T$ is the (j,i)-entry of AC, which is

column i of C

$$\underbrace{\begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jm} \end{bmatrix}}_{\text{row } j \text{ of } A} \begin{bmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{mi} \end{bmatrix} = a_{j1}c_{1i} + a_{j2}c_{2i} + \cdots + a_{jm}c_{mi}$$

the (i,j)-entry of $C^T A^T$ is the product of row i of C^T and column j of A^T, which is

(row j of A)^T

$$\underbrace{\begin{bmatrix} c_{1i} & c_{2i} & \cdots & c_{mi} \end{bmatrix}}_{(\text{column } i \text{ of } C)^T} \begin{bmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jm} \end{bmatrix} = c_{1i}a_{j1} + c_{2i}a_{j2} + \cdots + c_{mi}a_{jm}$$

The associative law allows using ABC to represent $(AB)C$ or $A(BC)$.
Power of square matrices: $A \in \mathcal{M}_{n\times n}$, $A^k = A\,A \cdots A$ (k times), and by convention, $A^1 = A$, $A^0 = I_n$.

Augmented matrix: If A and B are matrices with the same number of rows, then the augmented matrix $[A\ B]$ is the matrix whose columns are the columns of A followed by the columns of B.

Property: For any $P \in \mathcal{R}^{m\times n}$ and matrices A and B with n rows,
$P[A\ B] = [PA\ PB]$.

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}$, and $P = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$.

Then $[A\ B] = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 3 & 1 \end{bmatrix}$, $PA = P$, and $PB = \begin{bmatrix} 0 & 6 & 3 \\ 5 & -3 & 1 \\ -1 & 3 & 1 \end{bmatrix}$.

Also, $P[A\ B] = P\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 3 & 1 \end{bmatrix} = [PA\ PB]$ (you check).

Diagonal matrices: $A = [a_{ij}] \in \mathcal{M}_{n\times n}$ and $a_{ij} = 0$ for $i \neq j$, sometimes denoted by $A = \text{diag}[a_{11} \cdots a_{nn}]$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad AB = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Property: $A = \text{diag}[a_{11} \cdots a_{nn}]$, $B = \text{diag}[b_{11} \cdots b_{nn}] \Rightarrow$
$AB = \text{diag}[a_{11}b_{11} \cdots a_{nn}b_{nn}]$.

Symmetric matrices: $A = [a_{ij}] \in \mathcal{M}_{n\times n}$ and $a_{ij} = a_{ji}$, or $A = A^T$.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{bmatrix} = A^T \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq B^T$$

Property: For any $A \in \mathcal{M}_{n\times n}$, AA^T and A^TA are square and symmetric:
$(AA^T)^T = A^{TT}A^T = AA^T$ and $(A^TA)^T = A^TA^{TT} = A^TA$.

Homework Set for 2.1

Section 2.1: Problems 1, 3, 7, 9, 13, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 41, 47, 49