1.7 Linear Dependence and Linear Independence

Given a set of vectors, how to determine if there are any vectors that are linear combinations of other vectors?

Example

Consider the set $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ -8 \\ 13 \\ 8 \end{bmatrix}, \text{ and } \mathbf{u}_4 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

Is $\mathbf{u}_4$ a linear combination of others?

Can any vector be removed from $\mathcal{S}$ without affecting its span?

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -1 & 1 & -8 & 1 \\ 2 & -1 & 13 & -2 \\ 1 & -1 & 8 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Given a set of vectors, how to determine if there is any vector in it that is a linear combination of the other vectors?

Idea: in $\{\mathbf{u}_1, \ldots, \mathbf{u}_i, \ldots, \mathbf{u}_k\}$, if there exists any $\mathbf{u}_i$ that is a linear combination of other vectors, then there exists scalars $c_1, \ldots, c_i, \ldots, c_k$, that are not all zero, such that $c_1\mathbf{u}_1 + \cdots + c_i\mathbf{u}_i + \cdots + c_k\mathbf{u}_k = \mathbf{0}$.

Definition (L.D.)

A set of k vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ in $\mathcal{R}^n$ is called **linearly dependent** if there exist scalars $c_1, c_2, \cdots, c_k$, not all zero, such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

In this case, we also say that **the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly dependent.**

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Definition (L.I.)

A set of k vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ in $\mathcal{R}^n$ is called **linearly independent** if the only scalars $c_1, c_2, \cdots, c_k$ such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$$

are $c_1 = c_2 = \cdots = c_k = 0$. In this case, we also say that **the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent.**

Property

Any finite set $\mathcal{S} = \{\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ that contains the zero vector is L.D.

since $1 \cdot \mathbf{0} + 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + \cdots + 0 \cdot \mathbf{u}_k = \mathbf{0}$.

Property (Condition for L.D.)

The set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is **linearly dependent** if and only if $A\mathbf{x} = \mathbf{0}$ has a nonzero solution, where $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{bmatrix}$.

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0} \iff \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \mathbf{0}.$$

Example:

$$\mathcal{S}=\underbrace{\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\4\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}}_{A=[\ \cdots\ ]}$$

L.D. or L.I.? and which element(s), if any, can be linearly combined by others?

the augmented matrix of $A\mathbf{x}=\mathbf{0}$ is

$$\begin{bmatrix}1&1&1&1&0\\2&0&4&2&0\\1&1&1&3&0\end{bmatrix}\xrightarrow[\text{echelon form}]{\text{reduced row}}\begin{bmatrix}1&0&2&0&0\\0&1&-1&0&0\\0&0&0&1&0\end{bmatrix}$$

always zero column, redundant.

$\Rightarrow$ the general solution of $A\mathbf{x}=\mathbf{0}$ is $\begin{cases}x_1=-2x_3\\x_2=x_3\\x_3\quad\text{free}\\x_4=0\end{cases}$

setting $x_3=1$ leads to $-2\begin{bmatrix}1\\2\\1\end{bmatrix}+1\begin{bmatrix}1\\0\\1\end{bmatrix}+1\begin{bmatrix}1\\4\\1\end{bmatrix}+0\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

the augmented matrix of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix} \xrightarrow[\text{echelon form}]{\text{reduced row}} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

always zero column, redundant.

$\Rightarrow$ the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= x_3 \\ x_3 & \quad \text{free} \\ x_4 &= 0 \end{aligned}$$

setting $x_3 = 1$ leads to $-2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Conclusion: S is L.D., and the last vector is not a linear combination of others.

In general, the set is L.I. if and only if there is not any free variable.

Examples

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\}: \text{L.D.} \quad \mathcal{S}_2 = \left\{ \underbrace{\begin{bmatrix} -4 \\ 12 \\ 6 \end{bmatrix}, \begin{bmatrix} -10 \\ 30 \\ 15 \end{bmatrix}}_{2.5\mathbf{u}_1 = \mathbf{u}_2} \right\}: \text{L.D.}$$

not multiple

$$\mathcal{S}_3 = \left\{ \begin{bmatrix} -3 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}: \text{L.I.} \quad \mathcal{S}_4 = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \right\}$$

: L.D.

Let us define $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$.

- Questions
  - On which conditions will the columns of A be **linearly independent**?
  - 1. Does $A\mathbf{x} = \mathbf{0}$ always have a solution?
  - 2. Can $A\mathbf{x} = \mathbf{0}$ have more than one solution?
  - ➔ $A\mathbf{x} = \mathbf{0}$ should always have exactly one solution (i.e., $\mathbf{x}=\mathbf{0}$).
  - 3. Can $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions (for some nonzero $\mathbf{b}$)?
  - 4. Does $A\mathbf{x} = \mathbf{b}$ always have a solution?
  - ➔ $A\mathbf{x} = \mathbf{b}$ should always have no more than one solution.
  - 5. What is **rank** A? How about **nullity** A?
  - 6. What does the **reduced row echelon form** of A look like?

Theorem 1.8

The following statements about an $m \times n$ matrix A are equivalent:

(a) The columns of A are **linearly independent**.

(b) The equation $A\mathbf{x} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$ in $\mathcal{R}^m$.

(c) The **nullity** of A is zero.

(d) The **rank** of A is n, the number of columns of A.

(e) The columns of the **reduced row echelon form** of A are distinct **standard vectors** in $\mathcal{R}^m$.

(f) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.

(g) There is a **pivot position** in each column of A.

Proof

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(g) There is a **pivot position** in each column of A.

Proof (a) $\Leftrightarrow$ (f): by definition, as noted.

(b) $\Rightarrow$ (c): (b) $\Rightarrow$ $A\mathbf{x} = \mathbf{0}$ has the only solution $\mathbf{x} = \mathbf{0}$

$\Rightarrow$ general solution $A\mathbf{x} = \mathbf{0}$ has no free variables

$\Rightarrow$ nullity of A is zero.

(c) $\Rightarrow$ (d): rank A + nullity $A = n$.

(d) $\Rightarrow$ (e): every column of is A a pivot column

$\Rightarrow$ in the reduced row echelon form of A, every column has the form $[0 \cdots 0\ 1\ 0 \cdots 0]^T$, and the positions of 1's are different.

(e) $\Rightarrow$ (f): (e) $\Rightarrow$ the reduced row echelon form of A is $R = [\mathbf{e}_1\ \mathbf{e}_2\ \cdots\ \mathbf{e}_n]$, where $\mathbf{e}_i$ is the ith standard vector in $\mathcal{R}^m$, and clearly $A\mathbf{x} = \mathbf{0}$ has the only solution $\mathbf{x} = \mathbf{0}$.

(f) $\Rightarrow$ (b): suppose $\mathbf{u}$ and $\mathbf{v}$ are both solutions to $A\mathbf{x} = \mathbf{b}$

$\Rightarrow A(\mathbf{u} - \mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}$

$\Rightarrow \mathbf{u} - \mathbf{v}$ is the zero solution to $A\mathbf{x} = \mathbf{0}$ i.e., $\mathbf{u} = \mathbf{v}$.

(f) $\Leftrightarrow$ (g): no free variable in the solution of $A\mathbf{x} = \mathbf{0}$

Definition

A system of linear equations $A\mathbf{x} = \mathbf{b}$ is called **homogeneous** if $\mathbf{b} = \mathbf{0}$.

Properties of a homogeneous system of linear equations: $A\mathbf{x} = \mathbf{0}$

(1) always consistent, since $\mathbf{x}=\mathbf{0}$ is a solution;

(2) if it has nonzero solutions, then columns of A are L.D.;

(3) if number of variables > number of equations, then it has nonzero solutions, since free variables exist.

Example

$A\mathbf{x} = \mathbf{0}$ with $\begin{bmatrix} 1 & -4 & 2 & -1 & 2 \\ 2 & -8 & 3 & 2 & -1 \end{bmatrix}$

The reduced row echelon form of $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ is $\begin{bmatrix} 1 & -4 & 0 & 7 & -8 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \end{bmatrix}$

$$\Rightarrow \text{ the general solution is } \begin{cases} x_1 & = & 4x_2 & -7x_4 & +8x_5 \\ x_2 & & \text{free} & & \\ x_3 & = & & 4x_4 & -5x_5 \\ x_4 & & \text{free} & & \\ x_5 & & \text{free} & & \end{cases}$$

parametric representation of the general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4x_2 - 7x_4 + 8x_5 \\ x_2 \\ 4x_4 - 5x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 8 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

$S = \{\cdots\}$ is L.I. due to the positions of the 1's.

Property

When a **parametric representation** of the general solution to $A\mathbf{x} = \mathbf{0}$ is obtained by the method described in Section 1.3, the vectors that appear in the parametric representation are **linearly independent**.

Theorem 1.9

Vectors $\mathbf{u}_1, \mathbf{u}_2, ...\mathbf{u}_k$ in $\mathcal{R}^n$ are **linearly dependent** if and only if $\mathbf{u}_1 = \mathbf{0}$ or there exists an $i \geq 2$ such that $\mathbf{u}_i$ is a **linear combination** of the preceding vectors $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{i-1}$.

Proof

Theorem 1.9

Vectors $\mathbf{u}_1, \mathbf{u}_2, ...\mathbf{u}_k$ in $\mathcal{R}^n$ are **linearly dependent** if and only if $\mathbf{u}_1 = \mathbf{0}$ or there exists an $i \geq 2$ such that $\mathbf{u}_i$ is a **linear combination** of the preceding vectors $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{i-1}$.

Proof "if ($\Leftarrow$)": simple, you show it.

"only if ($\Rightarrow$)": Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is L.D., $\exists$ $c_1, c_2, \ldots, c_k$, not all zero, such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$.

Let $i = \max \{j: c_j \neq 0\}$.

Case (1): $i = 1 \Rightarrow c_1\mathbf{u}_1 = \mathbf{0} \Rightarrow \mathbf{u}_1 = \mathbf{0}$

Case (2): $i > 1 \Rightarrow c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_i\mathbf{u}_i = \mathbf{0}$

$$\Rightarrow \mathbf{u}_i = \frac{-c_1}{c_i}\mathbf{u}_1 - \frac{c_2}{c_i}\mathbf{u}_2 - \cdots - \frac{c_{i-1}}{c_i}\mathbf{u}_{i-1}$$

Properties of L.D. and L.I. Sets

(1) For a 1-vector set, $\{\mathbf{u}\}$ is L.I. as long as $\mathbf{u} \neq \mathbf{0}$. The set $\{\mathbf{0}\}$ is L.D.

(2) For a 2-vector set,

$$\begin{aligned} \{\mathbf{u}_1, \mathbf{u}_2\} \text{ is L.D.} \quad &\Longleftrightarrow \quad \mathbf{u}_1 = \mathbf{0}, \text{ or } \mathbf{u}_2 \text{ is a multple of } \mathbf{u}_1. \\ &\Longleftrightarrow \quad \text{one vector is a multiple of the other.} \end{aligned}$$

(3) Let $\mathcal{S} \neq = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a linear independent subset of $\mathcal{R}^n$, and $\mathbf{v}$ be in $\mathcal{R}^n$. Then

$$\mathbf{v} \notin \text{Span } \mathcal{S} \Leftrightarrow \mathcal{S} \cup \{\mathbf{v}\} \text{ is L.I.}$$

(4)Every subset of $\mathcal{R}^n$ containing more than n vectors must be L.D.

(5)No vector can be removed from a set $\mathcal{S} \subset \mathcal{R}^n$ without changing its span

$$\Rightarrow \mathcal{S} \text{ is L.I.}$$

Summary and Review:

Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n]$ be an m x n matrix.

Let $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

1) **Conditions for S to generate the whole space $\mathcal{R}^m$?**
2) **Conditions for S to be linearly independent?**

Rank of matrices, solutions for systems of linear equations, reduced row echelon forms, and linear independence

The rank of A	The number of solutions of Ax=b	The columns of A	The reduced row echelon form R of A
rank $A = m$	$A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ in $\mathcal{R}^m$.	The columns of A generate $\mathcal{R}^m$.	Every row of R contains a pivot position.
rank $A = n$	$A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$ in $\mathcal{R}^m$.	The columns of A are linearly independent.	Every column of R contains a pivot position.

Homework Set for 1.7

Section 1.7: Problems 1, 3, 7, 9, 13, 17, 21, 23, 25, 27, 29, 31, 39, 43, 49, 51, 53, 59, 61, 63, 69, 71, 75, 79

Chapter 1 in a nutshell

- Matrices, vectors.
- Linear combinations of vectors/ Matrix-Vector Product
- System of Linear Equations
  - Solution Set
  - Elementary row operations
  - (Reduced) row echelon forms
  - Gaussian Eliminations
- Rank, Nullity, basic variables, free variables
- Span of a set of vectors
  - Vectors in $\mathcal{R}^m$ that generate $\mathcal{R}^m$.
- Linear Independence
  - Smallest set of vectors that has the same span