1.4 Gaussian Elimination

Gaussian elimination: an algorithm for finding a (actually "the") **reduced row echelon form** of a matrix.

Original augmented matrix

A row echelon form

The reduced row echelon form

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Steps 1 – 4

Steps 5 - 6

Another Example

Original augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \rightarrow \cdots \rightarrow$$

A row echelon form

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced row echelon form

$$\rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

leading entries

pivot positions

pivot columns

Gaussian elimination: an algorithm for finding a (actually "the") **reduced** **row echelon form** of a matrix.

- Step 1:
  - Determine the *leftmost nonzero column*. This is a **pivot column**, and the topmost position (1st row) in this column is a **pivot position**.

pivot position

$$[\ A \mid \mathbf{b}\ ] = \left[\begin{array}{cccccc|c} 0 & \boxed{\begin{matrix} 0 \\ 1 \\ 6 \end{matrix}} & \begin{matrix} 2 \\ -1 \\ 0 \end{matrix} & \begin{matrix} -4 \\ 1 \\ -6 \end{matrix} & \begin{matrix} -5 \\ 3 \\ 5 \end{matrix} & \begin{matrix} 2 \\ 1 \\ 16 \end{matrix} & \begin{matrix} 5 \\ -1 \\ 7 \end{matrix} \end{array}\right]$$

pivot column

- Step 2:
  - In the **pivot column**, choose any nonzero entry in a row that is not ignored, and perform the appropriate **row interchange** to bring this entry into the **pivot position**.

interchange rows 1 and 2 $\longrightarrow$

$$\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 6 & 0 & -6 & 5 & 16 & 7 \end{array}\right]$$

- Step 3:
  - Add an appropriate multiple of the row containing the **pivot position** to each lower row in order to change each entry below the **pivot position** into zero.

$$\begin{array}{r} \times(-6) \\ \\ + \end{array}\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 6 & 0 & -6 & 5 & 16 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 6 & -12 & -13 & 10 & 13 \end{array}\right]$$

$$\left(\begin{array}{ccccccc l} 0 & 6 & 0 & -6 & 5 & 16 & 7 & \text{(row 3)} \\ 0 & -6 & 6 & -6 & -18 & -6 & 6 & (-6 \text{ times row } 1) \\ \hline 0 & 0 & 6 & -12 & -13 & 10 & 13 & \end{array}\right)$$

- Step 4:
  - Ignore the row that contains the **pivot position**. If there is a nonzero row that is not ignored, repeat steps 1-4 on the submatrix that remains.

pivot position

new pivot position

$$\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 6 & -12 & -13 & 10 & 13 \end{array}\right]$$

new pivot column

$$\begin{array}{r} \times(-3) \\ + \end{array}\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 6 & -12 & -13 & 10 & 13 \end{array}\right] \longrightarrow \left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 0 & 0 & 2 & 4 & -2 \end{array}\right]$$

$$\begin{array}{cccccccl} 0 & 0 & 6 & -12 & -13 & 10 & 13 & \text{(row 3)} \\ 0 & 0 & -6 & 12 & 15 & -6 & -15 & (-3 \text{ times row 2}) \\ \hline 0 & 0 & 0 & 0 & 2 & 4 & -2 & \end{array}$$

pivot position

$$\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 0 & 0 & \underbrace{2} & 4 & -2 \end{array}\right] : \text{a row echelon form}$$

previous
pivot positions

pivot column

- Step 5:
  - If the leading entry of the row is not 1, perform the appropriate scaling operation to make it 1. Then add an appropriate multiple of this row to every preceding row to change each entry above the pivot position into zero.

$$\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 0 & 0 & 2 & 4 & -2 \end{array}\right]$$

multiply row 3 by 1/2 →

$$\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

add (-3)× (row 3) to row 1
add 5 × (row 3) to row 2 →

$$\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 0 & -5 & 2 \\ 0 & 0 & 2 & -4 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

- Step 6:
  - If step 5 was performed using the first row, stop. Otherwise, repeat step 5 on the preceding row.

$$\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 0 & -5 & 2 \\ 0 & 0 & 2 & -4 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

multiply row 2 by 1/2 $\longrightarrow$

$$\left[\begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 0 & -5 & 2 \\ 0 & 0 & 1 & -2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

add row 2 to row 1 $\longrightarrow$

$$\underbrace{\left[\begin{array}{cccccc|c} 0 & 1 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right]}_{\text{the reduced row echelon form}}$$

steps 1-4: forward pass $\rightarrow$ a row echelon form steps 5-6: backward pass $\rightarrow$ a reduced row echelon form

Example

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}
$$

×(-1) + ×(-2) + ×3 +

$$
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
-1 & -2 & 1 & 2 & 3 & 6 \\
2 & 4 & -3 & 2 & 0 & 3 \\
-3 & -6 & 2 & 0 & 3 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & -1 & 6 & 6 & 15
\end{bmatrix}
\rightarrow
$$

×(-1)

$$
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & -1 & 6 & 6 & 15
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & 0 & 8 & 8 & 16
\end{bmatrix}
\rightarrow
$$

×(-2) +

$$
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\xrightarrow{\times(\frac{1}{4})}
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

+ +1 ×2 0-2

end of the forward pass,
and a row echelon form obtained

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \times(-1) \longrightarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a redundant equation is eliminated to become $0 = 0$

$$\begin{array}{rrrrrl} x_1 & +2x_2 & -x_3 & +2x_4 & +x_5 & =2 \\ -x_1 & -2x_2 & +x_3 & +2x_4 & +3x_5 & =6 \\ 2x_1 & +4x_2 & -3x_3 & +2x_4 & & =3 \\ -3x_1 & -6x_2 & +2x_3 & & +3x_5 & =9 \end{array}$$

general solution: $\begin{cases} x_1 = -5 - 2x_2 + x_5 \\ x_2 \quad \text{free} \\ x_3 = -3 \\ x_4 = 2 - x_5 \\ x_5 \quad \text{free} \end{cases}$

$$\Leftrightarrow \begin{array}{rrrrl} x_1 + 2x_2 & + & & -x_5 & = -5 \\ & x_3 & & & = -3 \\ & & x_4 & + x_5 & = 2 \end{array}$$

$$\text{Solution set} = \left\{ \begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} : x_2, x_5 \in \mathcal{R} \right\}$$

Question:

$m \times (n+1)$

Let $[A\ \mathbf{b}]$ be an augmented matrix which has a reduced row echelon form $[R\ \mathbf{c}]$. $m \times (n+1)$

Is R also a reduced row echelon form of A?

$m \times n$ $m \times n$

Rank and Nullity

Definitions

1. The **rank** of an $m \times n$ matrix A, denoted by rank A, is defined to be the number of nonzero rows in the reduced row echelon form of A.
2. The **nullity** of A, denoted by nullity A, is defined to be $n - \text{rank } A$.

Let's first calculate rank and nullity of the augmented matrix and then the coefficient matrix presented in the previous page and try to figure out what these mean.

$$\overbrace{\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}}^{A \qquad\qquad \mathbf{b}} \longrightarrow \overbrace{\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}^{R \qquad\qquad \mathbf{c}}$$

$\text{rank} \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = 3$ $\qquad$ $\text{rank } A = 3$

$\text{nullity} \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = 6-3=3$ $\qquad$ $\text{nullity } A = 5-3=2$

$$\left[\begin{array}{ccccc|c} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad 2 \quad -1 \quad 2 \quad 1}}^{A}} & \overbrace{}^{\mathbf{b}} \\ 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad 2 \quad 0 \quad 0 \quad -1}}^{R}} & \overbrace{}^{\mathbf{c}} \\ 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{rank} \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \text{rank} \begin{bmatrix} R & \mathbf{c} \end{bmatrix} = 3 \qquad \text{rank } A = \text{rank } R = 3$$

$$\text{nullity} \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \text{nullity} \begin{bmatrix} R & \mathbf{c} \end{bmatrix} = 3 \qquad \text{nullity } A = \text{nullity } R = 2$$

$$\begin{array}{l} x_1 + 2x_2 - x_3 + 2x_4 + 5x_5 = 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 = 6 \\ 2x_1 + 4x_2 - 3x_3 + 2x_4 = 3 \\ -3x_1 - 6x_2 + 2x_3 + 3x_5 = 9 \end{array} \longrightarrow \begin{array}{l} x_1 + 2x_2 - x_5 = -5 \\ x_3 = -3 \\ x_4 + x_5 = 2 \\ 0 = 0 \end{array} \longrightarrow \begin{array}{l} x_1 = -5 - 2x_2 + x_5 \\ x_2 \ \text{free} \\ x_3 = -3 \\ x_4 = 2 - x_5 \\ x_5 \ \text{free} \end{array}$$

Interpretations:

1. The rank of [A **b**] is the number of “useful” equations in the system of linear equations $A\mathbf{x} = \mathbf{b}$ (i.e., the number of nonzero rows in [R **c**]).

2. The rank of A represents the number of basic variables while the nullity of A is the number of free variables.

nullity A = n - rank A

$$\left[\;\overbrace{\begin{matrix}1 & 2 & -1 & 2 & 1\\ -1 & -2 & 1 & 2 & 3\\ 2 & 4 & -3 & 2 & 0\\ -3 & -6 & 2 & 0 & 3\end{matrix}}^{A}\;\;\overbrace{\begin{matrix}2\\ 6\\ 3\\ 9\end{matrix}}^{\mathbf{b}}\;\right] \longrightarrow \left[\;\overbrace{\begin{matrix}1 & 2 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\end{matrix}}^{R}\;\;\overbrace{\begin{matrix}-5\\ -3\\ 2\\ 0\end{matrix}}^{\mathbf{c}}\;\right]$$

$$\operatorname{rank}\begin{bmatrix} A & \mathbf{b}\end{bmatrix} = \operatorname{rank}\begin{bmatrix} R & \mathbf{c}\end{bmatrix} = 3 \qquad \operatorname{rank} A = \operatorname{rank} R = 3$$

$$\operatorname{nullity}\begin{bmatrix} A & \mathbf{b}\end{bmatrix} = \operatorname{nullity}\begin{bmatrix} R & \mathbf{c}\end{bmatrix} = 3 \qquad \operatorname{nullity} A = \operatorname{nullity} R = 2$$

Interpretations:

1. The rank of [A $\mathbf{b}$] is the number of "useful" equations in the system of linear equations $A\mathbf{x} = \mathbf{b}$ (i.e., the number of nonzero rows in [R $\mathbf{c}$]).

2. The rank of A represents the number of basic variables while the nullity of A is the number of free variables.

Question:

Will rank [A $\mathbf{b}$] always be equal to rank A?

Rank and Nullity

Definitions

1. The **rank** of an $m \times n$ matrix A, denoted by rank A, is defined to be the number of nonzero rows in the reduced row echelon form of A.
2. The **nullity** of A, denoted by nullity A, is defined to be $n - \text{rank } A$.

Proposition

If $A\mathbf{x} = \mathbf{b}$ is the matrix form of a consistent system of linear equations, then:
(a) the number of basic variables in a general solution to the system equals the rank of A; and
(b) the number of free variables in a general solution to the system equals the nullity of A.

On the consistency of a system of linear equations

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

- Question
  - Can we determine whether a system of linear equation is **consistent** by means of some linear algebra terms we have learned, for example, "**reduced row echelon form**," "**augmented matrix,**" "**linear combination,**" "**rank,**" etc.?

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad [\ A \mid \mathbf{b}\ ] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Theorem 1.5 (Test of consistency)

Let $A \in \mathcal{M}_{m\times n}$ and $\mathbf{b} \in \mathcal{R}^m$. Then the following conditions are equivalent:

(a) The matrix equation $A\mathbf{x} = \mathbf{b}$ is **consistent**.

(b) The vector $\mathbf{b}$ is a **linear combination** of the columns of A.

(c) The **reduced row echelon form** of the **augmented matrix** $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ has no row of the form $\begin{bmatrix} 0 & 0 & \cdots & 0 & d \end{bmatrix}$ where $d \neq 0$.

Proof

(a) $\Leftrightarrow$ (b): (b) means $\exists\ \mathbf{v} = [\ v_1\ v_2 \cdots v_n\ ]^T$ such that

$$A\mathbf{v} = [\ \mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n\ ]\mathbf{v} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \cdots + v_n\mathbf{a}_n = \mathbf{b}$$

thus $\mathbf{v}$ is a solution, and $A\mathbf{x} = \mathbf{b}$ is consistent.

(a) $\Leftrightarrow$ (c): Explained in Section 1.3.

$$[A \mid b] \longrightarrow [R \mid c]$$

Theorem 1.5 (Test of consistency)

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(d) The **rank** of the **coefficient matrix** is equal to the **rank** of the **augmented matrix**, i.e., rank A = rank $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$.

$$\left[\begin{array}{c|c} A & \underline{b} \end{array}\right] \rightarrow \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad}^{R} & \overbrace{\quad}^{c} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

rank $A = 2$

rank$[A\ \underline{b}] = 3$

Proof

(c) $\Leftrightarrow$ (d): Let $[\,R \;\; \mathbf{c}\,]$ be the reduced row echelon form of $[\,A \;\; \mathbf{b}\,]$, then from the definition of the reduced echelon form, R is the reduced row echelon form of A.

Section 1.4 (Review)

- Gaussian Elimination
  - Steps 1~4 make the augmented matrix into a row echelon form.
  - Steps 5~6 further transform it into a reduced row echelon form.
- Rank, Nullity
- Test for consistency

Homework Set for 1.4

Section 1.4: Problems 3, 7, 11, 15, 21, 25, 29, 37, 41, 53, 55, 57, 59, 63, 67, 71.

An analogy: Solving a Rubik’s cube

A randomly scrambled Rubik’s cube

$\to \cdots \to$

The first face solved

$\to \cdots \to$

All six faces solved