# 1.3 Systems of Linear Equations Examples of **Systems of Linear Equations**



Today, we will learn the general form of **systems of linear equations** and learn to solve any of such systems systematically.

The materials will be closely related to **matrices** and **vectors** we introduced in the previous sections.



A solution of a system of linear equations in the variables  $x_1, x_2, ..., x_n$  is a vector  $\begin{bmatrix} s_2 \\ \vdots \\ s_n \end{bmatrix}$  in  $\mathcal{R}^n$  such that every equation in the system is satisfied when each  $x_i$  is replaced by  $s_i$ .  $\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \end{array}$  $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$ For example,  $\begin{vmatrix} 2\\5\\1 \end{vmatrix} \in \mathcal{R}^3$  is a solution of the system  $\begin{array}{rcrcrcrcr} 2x_1 - 3x_2 + x_3 & = & -10 \\ x_1 & + x_3 & = & 3 \end{array}$ Note:  $\begin{vmatrix} -4 \\ 3 \\ 7 \end{vmatrix}$  is also a solution!

• The set of all solutions of a system of linear equations is called the solution set.

 $\mathscr{R}^2$  denotes the set of all 2-component vectors.





Consider any system of linear equations with m = n = 2. That is, two linear equations in two variables (two lines in  $\mathcal{R}^2$ ).

Your learned from high school algebra that the solution set of such a system will fall into one of the following three categories.



How about a system of linear equations with larger *m* and/or *n*?

**Claim**: Every system of linear equation has no solution, exactly one solution, or infinitely many solutions. (Will be proved subsequently)

- A system of linear equations is called **consistent** if it has one or more solutions.
- A system of linear equations is called inconsistent if its solution set is empty.

|                                   | $\begin{array}{rcrcrcrc} 3x_1 + x_2 & = & 10 \\ x_1 - 3x_2 & = & 0 \end{array}$ | $3x_1 + x_2 = 10 6x_1 + 2x_2 = 20$  | $\begin{array}{rcl} 3x_1 + x_2 &=& 10 \\ 6x_1 + 2x_2 &=& 0 \end{array}$ |
|-----------------------------------|---|---|---|
| Solution set                      | $\left\{ \left[ \begin{array}{c} 3\\1 \end{array} \right] \right\}$             | $\left\{ \left[ \begin{array}{c} 3\\1 \end{array} \right] + t \left[ \begin{array}{c} -1\\3 \end{array} \right] : \forall t \in \mathcal{R} \right\}$ | $\{\}, \text{ or } \phi$  |
| Consistent<br>or<br>Inconsistent? | Consistent  | Consistent  | Inconsistent  |

• Two systems of linear equations are called **equivalent** if they have exactly **the same solution set**.



#### • Elementary Row Operations

- Simple operations defined on a system of linear equations that do not change the solution set of the system of linear equations (i.e., they are all **equivalent**)
- Used for solving systems of linear equations



Three types of elementary row operations

#### 1. Interchange

$$\begin{cases} 3x_1 + x_2 &= 10 \\ x_1 + 0x_2 &= 3 \end{cases} \longrightarrow \begin{cases} x_1 + 0x_2 &= 3 \\ 3x_1 + x_2 &= 10 \end{cases}$$

### 2. Scaling

$$\begin{cases} 3x_1 + x_2 &= 10 \\ 10x_1 + 0x_2 &= 30 \end{cases} (\times \frac{1}{10}) \implies \begin{cases} 3x_1 + x_2 &= 10 \\ x_1 + 0x_2 &= 3 \end{cases}$$

### 3. Row addition

$$\begin{cases} 3x_1 + x_2 = 10 & (\times 3) \\ x_1 - 3x_2 = 0 & (+) & (+) & (\times 3) \\ 10x_1 + 0x_2 = 30 & (+) & (+) & (\times (-3)) \\ 3x_1 + x_2 = 10 & (+) & (+) & (+) & (\times (-3)) \\ 0x_1 + x_2 = 1 & (+) &$$

Equivalent systems of linear equations: having the same solution set. An elementary row operation taken on a system of linear equations will result in another equivalent system of linear equations Elementary row operations for solving systems of linear equations:

$$(-2) \times ((x_1 - 2x_2 - x_3 = 3)) \times (-3) \qquad x_1 - 2x_2 - x_3 = 3 3x_1 - 6x_2 - 5x_3 = 3 + 2x_1 - x_2 + x_3 = 0 \qquad \Rightarrow \qquad x_1 - 2x_2 - x_3 = 3 + 3x_2 + 3x_3 = -6 \qquad \Leftrightarrow \qquad x_1 - 2x_2 - x_3 = 3 + 3x_2 + 3x_3 = -6 \qquad \Leftrightarrow \qquad x_1 - 2x_2 - x_3 = 3 + 3x_2 + 3x_3 = -6 \qquad \Leftrightarrow \qquad x_1 - 2x_2 - x_3 = 3 + 3x_2 + 3x_3 = -6 \qquad \Leftrightarrow \qquad x_1 - 2x_2 - x_3 = 3 \\ + 3x_2 + 3x_3 = -6 \qquad \Rightarrow \qquad x_1 - 2x_2 - x_3 = 3 \\ + 3x_2 + 3x_3 = -6 \qquad \Rightarrow \qquad x_1 - 2x_2 - x_3 = 3 \\ + 3x_2 + 3x_3 = -6 \qquad \Rightarrow \qquad x_1 - 2x_2 - x_3 = 3 \\ + 3x_2 + 3x_3 = -6 \qquad \Rightarrow \qquad x_1 - 2x_2 - x_3 = 3 \\ + 3x_2 + 3x_3 = -6 \qquad \Rightarrow \qquad x_1 - 2x_2 - x_3 = -6 \\ + 3x_2 + 3x_3 = -6 \qquad \Rightarrow \qquad x_1 - 2x_2 - x_3 = -6 \\ + 3x_2 + 3x_3 = -6 \qquad \Rightarrow \qquad x_1 - 2x_2 - x_3 = -6 \\ + 3x_2 + 3x_3 = -6 \qquad \Rightarrow \qquad x_1 - 2x_2 - x_3 = -6 \\ + 3x_2 + 3x_3 = -6 \\ + 3x_3 + 3x_3 = -6 \\$$

$$\begin{array}{rcl} x_1 - 2x_2 - x_3 &=& 3\\ 3x_2 + 3x_3 &=& -6\\ (-2x_3 &=& -6) \times (-1/2) \end{array} \Leftrightarrow \begin{array}{rcl} x_1 - 2x_2 - x_3 &=& 3\\ \Rightarrow & & & 3x_2 + 3x_3 &=& -6\\ (-3) \times ((x_3 &=& 3)) \times 1 \end{array} \Leftrightarrow$$

• In general, a system of linear equation  

$$a_{11}x_{1} + a_{12}x_{2} + \dots + a_{1n}x_{n} = b_{1}$$

$$a_{21}x_{1} + a_{22}x_{2} + \dots + a_{2n}x_{n} = b_{2}$$
:  
can be rewritten as  

$$A\mathbf{x} = \mathbf{b}$$
where  

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$
is called the **coefficient matrix**.  

$$\begin{pmatrix} \mathbf{x} = \begin{bmatrix} x_{1} \\ x_{2} \\ \vdots \\ x_{n} \end{bmatrix}$$
is the variable vector  

$$\begin{bmatrix} A \mid \mathbf{b} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{mn} \end{bmatrix}$$
is called the **augmented matrix**.

## **Elementary Row Operations**

# Definition

Any one of the following three operations performed on a matrix is called an **elementary row operation**:

1. Interchange any two rows of the matrix. (interchange operation)

2. Multiply every entry of some row of the matrix by the same **nonzero scalar**. (scaling operation).

3. Add a multiple of one row of the matrix to another row. (row addition operation)

Properties:

1. Every elementary row operations are reversible. For interchange operation it is obvious, for scaling operation multiply the inverse of the nonzero constant, and for row addition operation add the negative multiple of the row to the other row.

2. equivalent  

$$A\mathbf{x} = \mathbf{b}$$
  $\Leftrightarrow$   $A'\mathbf{x} = \mathbf{b}'$   
 $\|$   
 $[A \ \mathbf{b}]$  elementary row operations  $[A' \ \mathbf{b}']$ 

Elementary row operations taken on systems of linear equations as well as on the corresponding **augmented matrices**:

$$(-2) \times ((x_1 - 2x_2 - x_3 = 3)) \times (-3) \qquad x_1 - 2x_2 - x_3 = 3 3x_1 - 6x_2 - 5x_3 = 3 \qquad + \qquad \Leftrightarrow \qquad -2x_3 = -6 \qquad \Leftrightarrow + 2x_1 - x_2 + x_3 = 0 \qquad 3x_2 + 3x_3 = -6 \qquad \Leftrightarrow$$

$$\begin{pmatrix} (-2) \times \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{\times (-3)} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & -2 & 6 \\ 0 & 3 & 3 & -6 \end{bmatrix} \Leftrightarrow$$

$$x_{1} - 2x_{2} - x_{3} = 3 \\ 3x_{2} + 3x_{3} = -6 \\ (-2x_{3} = -6) \times (-1/2)$$

$$\begin{cases} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & -2 & 6 \end{cases}$$

$$x_{1} - 2x_{2} - 1 = 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 1 & 3 \end{cases}$$

$$x_{1} - 2x_{2} = 6 \\ (3x_{2} = -15) \times (-1/3) \\ x_{3} = 3 \end{cases}$$

$$x_{1} - 2x_{2} = 6 \\ (x_{2} = -5) \times 2 \\ x_{3} = 3 \end{cases}$$

$$x_{1} - 2x_{2} = 6 \\ (x_{2} = -5) \times 2 \\ x_{3} = 3 \end{cases}$$

$$x_{1} = -4 \\ (x_{2} = -5) \times 2 \\ x_{3} = 3 \end{cases}$$

$$x_{3} = 3 \qquad x_{3} = 3$$

$$x_{3} = 3 \qquad x_{3} = 3$$

### **Row Echelon Form and Reduced Row Echelon Form**

### Definition

A matrix is said to be in **row echelon form** if it satisfies the following three conditions:

1. Each nonzero row lies above every zero row.

2. The **leading entry** of a nonzero row lies in a column to the right of the column containing the **leading entry** of **any preceding row**.

3. If a column contains the **leading entry** of some row, then all entries of the column below the leading entry are 0. (Actually condition 3 is implied by 2.)

## Definition

A matrix is said to be in **reduced row echelon form** if it satisfies the following three conditions:

1-3. The matrix is in **row echelon form**.

4. If a column contains the **leading entry** of some row, then **all the other entries** of that column are 0.

5. The leading entry of each nonzero row is 1.

Examples of reduced row echelon form:





Example:  $A = \begin{bmatrix} 1 & 0 & 0 & 6 & 3 & 0 \\ 0 & 0 & 1 & 5 & 7 & 0 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ 

not a row echelon form

|     | 1 | 7 | 2 | -3  | 9 | 4 |
|-----|---|---|---|-----|---|---|
|     | 0 | 0 | 1 | 4   | 6 | 8 |
| B = | 0 | 0 | 0 | (2) | 3 | 5 |
|     | 0 | 0 | 0 | Ŏ   | 0 | 0 |
|     | 0 | 0 | 0 | 0   | 0 | 0 |

row echelon form, but not a reduced row echelon form

### **Observation**

A system of linear equations is **easily solvable** if its **augmented matrix** is in **(reduced) row echelon form**.

Example:  $x_1 = -4$   $x_2 = -5$ : special reduce row echelon form [ $I \mathbf{b}'$ ]  $x_3 = 3 \iff \mathbf{x} = \mathbf{b}'$ , unique solution  $\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ 

### Observation

A system of linear equations is **easily solvable** if its **augmented matrix** is in **(reduced) row echelon form**.

Example:  $x_1 - 3x_2 + 2x_4 = 7$ basic  $x_3 + 6x_4 = 9$  free  $x_1 = 7 + 3x_2$  $x_2$  free  $x_3 = 9 - 6x_4$  $x_1 = 7 + 3x_2 - 2x_4$ variables  $x_4$  free 0 = 0 $x_5 = 2$  $\begin{bmatrix} 1 & -3 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & 6 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ general solution:  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7+3x_2-2x_4 \\ x_2 \\ 9-6x_4 \\ x_4 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 9 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix}$ 

: with free variables there are infinitely many solutions.

### Observation

A system of linear equations is **easily solvable** if its **augmented matrix** is in **(reduced) row echelon form**.

Whenever an augmented matrix contains a row in which **the only nonzero entry lies in the last column**, the corresponding system of linear equations has **no solution** 

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{array}{c} x_1 & -3x_3 = 0 \\ x_2 + 2x_3 = 0 \\ \vdots \text{ inconsistent system} \\ \text{of linear equations} \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{array}$$

meaning: in the original system of linear equations, there are mutually contradictory equations.

Which of the followings are in **row echelon form**? Which are in **reduced row echelon form**?  $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix} \quad \leftrightarrow \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & -2 & 6 \\ 0 & 3 & 3 & -6 \end{bmatrix} \leftrightarrow$  $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & -2 & 6 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix} \leftrightarrow$  $\begin{vmatrix} 1 & -2 & 0 & 6 \\ 0 & 3 & 0 & -15 \\ 0 & 0 & 1 & 3 \end{vmatrix} \leftrightarrow \begin{vmatrix} 1 & -2 & 0 & 6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{vmatrix} \leftrightarrow$  $\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ 

### **Row Echelon Form and Reduced Row Echelon Form**

Questions

1. Can **any** matrix be transformed into a matrix in **row echelon form** using a finite number of **elementary row operations**?

2. Can **any** matrix be transformed into a matrix in **reduced row echelon form** using a finite number of **elementary row operations**?

3. Can **any** matrix be transformed into a **unique** matrix in **row echelon form** using a finite number of **elementary row operations**?

4. Can **any** matrix be transformed into a **unique** matrix in **reduced row echelon form** using a finite number of **elementary row operations**?

#### **Theorem 1.4** (To be proved in Sections 1.4 and 2.3)

Every matrix can be transformed into **one** and **only one** in **reduced echelon form** by means of a sequence of **elementary row operations**.

A general procedure for solving  $A\mathbf{x} = \mathbf{b}$ :

- 1. Write the augmented matrix  $[A \mathbf{b}]$  of the system.
- 2. Find the reduced row echelon form [*R* c] of [*A* b] (with a finite number of elementary row operations).
- 3. (a) If [*R* c] contains a row in the form of [0 0 ... 0 1], then *A*x=b has no solution.

(b) Otherwise, it has at least one solution. In this case, write the system of linear equations corresponding to the matrix [R c], and solve this system for the basic variables in terms of the free variables to obtain a general solution of Ax=b.

### 1.3 Systems of Linear Equations

- In this section, we have defined the following terms:
  - Linear equation
  - System of linear equations
    - Solution
    - Solution set
    - **Consistent** and **inconsistent** system of linear equations
    - Equivalent systems of linear equations
    - **Coefficient matrix** and **augmented matrix** of systems of linear equations.
    - Basic variables and free variables
  - Elementary row operations
  - Row echelon form and reduced row echelon form.

#### **Homework Set for 1.3**

Section 1.3: Problems 1, 3, 5, 7-22, 55, 57, 59, 61, 63, 65, 67, 69, 71, 73, 75